Indian Statistical Institute

Analysis II (HW - 2) Date: February 15, 2022 Instructor : Jaydeb Sarkar

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NOTE: (i) $B[a,b]$ = the set of all bounded real-valued functions on $[a,b]$. (ii) $R[a,b]$ = the set of all Riemann integrable functions on $[a,b]$. (iii) $C[a,b]$ = the set of all continuous functions on $[a,b]$. (iv) $C[a,b]$ = the set of all partitions on $[a,b]$.

(1) Give an example to show that the composition of Riemann integrable functions need not be Riemann integrable.

(2) Let $f \in C[0,1]$ and suppose that $f(x) \neq 0$ for all $x \in (0,1)$. If

$$f(x)^2 = 2\int_0^x f(t)dt \qquad (\forall x \in [0,1]),$$

then prove that $f(x) = x$ for all $x \in [0,1]$.

(3) Let $f \in C[a,b]$. If $\int_a^x f = \int_x^b f$ for all $x \in [a,b]$, then prove that $f \equiv 0$.

(4) Compute the derivative (if it exists) of the function $x \mapsto \int_0^x \sqrt{t^2+4}\,dt$.

(5) Prove that $\ln x \leq 2(\sqrt{x}-1)$ for all $x \geq 1$. [Hint: $\frac{1}{x} \leq \frac{1}{\sqrt{x}}$ for all $x \geq 1$.]

(6) Let $f \in R[a,b]$. Consider the function $F(x) = \int_a^x f$, $x \in [a,b]$. Prove that

(i) if $f \geq 0$ on $[a,b]$, then F is monotonic increasing.

(ii) if $f \leq 0$ on $[a,b]$, then F is monotonic decreasing.

(7) Prove that $\int_{-1}^x sgn = |x| - 1$ for all $x \in \mathbb{R}$ (here sgn is the sign function).

[Note: Hence $\int_a^x f$ can exist for all $x \in [a,b]$ even when f has no antiderivative.]

(8) Suppose $f \in C[a,b]$ and $f(x) = \int_a^x f$. Prove that $f \equiv 0$.

(9) Let $f(x) = [x]$ for all $x \in [0, 3.5]$.

(a) Prove that $f \in R[0, 3.5]$.

(b) Evaluate $\int_0^{3.5} f$.

(c) Prove that $\int_0^{3.5} f$ can not be evaluated by the fundamental theorem of calculus.

(10) Suppose $F \in C[a,b]$ is differentiable on $[a,b] \setminus X$, where X is a finite set, and let $f \in R[a,b]$. If $F'(x) = f(x)$, $x \in [a,b] \setminus X$, then prove that

$$\int_a^b f = F(b) - F(a).$$

(11) Compute (i) $\frac{d}{dx}\Big(\int_{-x}^x e^{t^2}\,dt\Big)$, (ii) $\frac{d}{dx}\Big(\int_0^{x^2} \sin(t^2)\,dt\Big)$.

(12) Let $f \in C[a,b]$. If $\int_a^r f = 0$ for all rational r in $[a,b]$, then prove that $f \equiv 0$.

(13) Let $f, g \in C[a,b]$, and suppose $\int_a^b f = \int_a^b g$. Prove that $f(\zeta) = g(\zeta)$ for some $\zeta \in [a,b]$.

(14) Let $f \in C[0,1]$. Prove that

$$\int_0^1 \Big(\int_0^x f(t)dt\Big)dx = \int_0^1 (1-x)f(x)dx.$$

(15) (Weighted mean value theorem for integrals) Let $f \in C[0,1]$ and $g \in R[a,b]$. Suppose that g does not change sign in $[a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b fg = f(c)\int_a^b g.$$

(16) Let $f \in C[-1,1]$ and suppose

$$\int_{-x}^x f = 0,$$

for all $x \in (0,1]$. Prove that f is an odd function.