Indian Statistical Institute

Analysis II (HW - 3) Date: February 28, 2022 Instructor : Jaydeb Sarkar

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(1) Examine the convergence of

$$(i) \int_0^1 \frac{dx}{\sqrt{1-x^3}},\ (ii) \int_0^1 e^{-mx} x^n dx,\ (iii) \int_0^1 \Big(\log \frac{1}{x}\Big)^m dx,$$

where m and n are in $\mathbb{Z}$.

(2) Suppose $m, n \in \mathbb{Z}$. Prove that $\int_0^1 x^m (1-x)^n dx$ exists if and only if $m > 1$ and $n > 1$.

(3) Determine all natural number m and n for which $\int_0^1 f$ converges, where

$$f(x) = \frac{1}{(\sin x)^m (1-x)^n}.$$

(4) (Cauchy–Schwarz Inequality for Improper Integrals:) Let $f, g \in C[a, b)$. Suppose both f and g are improperly integrable on $[a, b)$. Prove that fg is improperly absolutely integrable on $[a, b)$ and

$$\Big(\int_a^b |fg|\Big)^2 \leq \Big(\int_a^b f^2\Big)\Big(\int_a^b g^2\Big).$$

(5) Let f, g be improperly integrable functions on $[a, \infty)$, and let $\alpha, \beta \in \mathbb{R}$. Prove that $\alpha f + \beta g$ is improperly integrable on $[a, \infty)$, and

$$\int_a^\infty (\alpha f + \beta g) = \alpha \int_a^\infty f + \beta \int_a^\infty g.$$

(6) Prove that

$$\lim_{c \to 1^-} \int_{-c}^{c} \frac{x}{(x^2-1)^2} dx = 0.$$

Does this imply that $\int_{-1}^1 \frac{x}{(x^2-1)^2} dx$ converges to 0? Use your favourite programming/technology to plot the graph of $x \mapsto \frac{x}{(x^2-1)^2}$ over $(-1, 1)$ and figure out what is going on.

(7) Let $f : [-1, 1] \to \mathbb{R}$ be an even, continuous function, and suppose $f(0) \neq 0$. Prove that

$$\int_{-1}^0 \frac{f(x)}{x} dx \text{ and } \int_0^1 \frac{f(x)}{x} dx,$$

diverge, and conclude that we cannot define

$$\int_{-1}^1 \frac{f(x)}{x} dx.$$

Also compute

$$\lim_{\epsilon \to 0} \Big(\int_{-1}^{-\epsilon} \frac{f(x)}{x} dx + \int_{\epsilon}^1 \frac{f(x)}{x} dx \Big).$$

(8) Prove that $\int_0^1 \frac{(\sin x)^m}{x^n} dx$ converges if and only if $m < 1 + n$.

(9) Let $a \in \mathbb{R}$, and let $f : [a, \infty) \to \mathbb{R}$ be an increasing function. Suppose $f \in R[a, b]$ for all $b > a$. Prove that f is improperly integrable on $[a, \infty)$ if and only if $f(x) = 0$ for all $x \geq a$.