Indian Statistical Institute

Analysis II (HW - 4) Date: March 7, 2022 Instructor : Jaydeb Sarkar

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(1) Let $p > 0$. Prove that $\int_1^\infty x^{-px} dx$ converges and $\int_1^\infty x^{-\frac{p}{x}} dx$ diverges. Show that the same for $\int_0^1 x^{-px} dx$ and $\int_0^1 x^{-\frac{p}{x}} dx$.

(2) Does $\int_a^\infty f$ exists implies $\int_r^R f \to 0$ as $r, R \to \infty$?

(3) Determine whether the integral $\int_0^1 x^n \ln x dx$, $n \in \mathbb{Z}$, converges.

(4) Prove that the improper integral

$$\int_0^1 \frac{1}{x}\left(\sin\frac{1}{x}\right) dx,$$

converges conditionally.

(5) Determine whether the integral $\int_0^\infty \frac{\sin x}{x^p} dx$ converges.

(6) Prove by induction that

$$\int_0^\infty t^n e^{-t} dt = n! \qquad (n \in \mathbb{N}).$$

(7) Discuss the convergence of $\int_0^\infty \frac{\sin x^m}{x^n} dx$.

(8) Determine whether the following integrals converge:

$$(i) \int_0^\infty \frac{1}{\sqrt{x} + e^x} dx, \ (ii) \int_{-\infty}^\infty \cos \pi x \, dx, \ (iii) \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx.$$

(9) Suppose f is absolutely integrable on $[1, \infty)$. Prove that

$$\lim_{n\to\infty} \int_1^\infty f(x^n) dx = 0.$$

(10) Let f and g be positive and continuous on $[1, \infty)$, and suppose $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists. Prove that

$$\lim_{x\to\infty} \frac{\int_x^\infty f}{\int_x^\infty g} = \lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

(11) Prove that $\int_2^\infty \frac{\cos x}{\log x} dx$ is conditionally convergent.

(12) Use integral test to determine the convergency of

$$(i) \sum_{n=2}^\infty \frac{1}{n \log n}, \quad (ii) \sum_{n=1}^\infty n e^{-n^2}, \quad (iii) \sum_{n=1}^\infty \frac{1}{n^2 + n}.$$

(13) Let $\int_a^\infty f$ converges. If $\lim_{x\to\infty} f(x) = l$ exists, then prove that $l = 0$.

(14) Product of improperly integrable functions may not be improperly integrable: Define

$$f(x) = \begin{cases} n & \text{if } x \in [n, n + n^{-\frac{5}{2}}), \ n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\int_1^\infty f$ converges, whereas $\int_1^\infty f^2$ diverges.