Indian Statistical Institute

Analysis II (HW - 4) Date: March 18, 2022 Instructor : Jaydeb Sarkar

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(1) Compute the pointwise limit of $\{f_n\}$, where

$$f_n(x) = \frac{x^n}{1+x^n} \qquad (n \geq 1, x \geq 0).$$

Also prove that $\{f_n\}$ converges uniformly on $[0, \epsilon]$ for all $0 < \epsilon < 1$, but not on $[0, 1]$.

(2) Prove that the sequence $\{\frac{nx}{1+n^3x^2}\}$ converges uniformly on $[0, 1]$.

(3) Prove that the sequence $\{\frac{n^2x}{1+n^3x^2}\}$ is not uniformly convergent on $[0, 1]$.

(4) Let $f_n(x) = \frac{1}{n}\sin nx$ for all $x \in \mathbb{R}$ and $n \geq 1$. Prove that $f_n \to 0$ uniformly on $\mathbb{R}$.

(5) Let $f(x) = \frac{1}{x} - \frac{x}{n}$, $x \in (0, 1]$. Prove that $\{f_n\}$ converges uniformly on $(0, 1]$ to the limit function $f(x) = \frac{1}{x}$, $x \in (0, 1]$.

(6) Let $p, q > 0$, and let

$$f_n(x) = \frac{x^p}{n+x^q} \qquad (n \geq 1, x \geq 0).$$

Prove that $f_n \to 0$ uniformly on $[0, \infty)$ if and only if $p < q$.

(7) Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions on $S \subseteq \mathbb{R}$, and suppose $f_n \to f$ and $g_n \to g$ uniformly on S. Prove that $f_n \pm g_n \to f \pm g$ uniformly on S.

(8) Let $\{f_n\}$ be a sequence of functions on $S \subseteq \mathbb{R}$, and suppose $f_n \to f$ uniformly on S. If $\{\frac{1}{f_n}\}$ is uniformly bounded on S, then prove that $\frac{1}{f_n} \to \frac{1}{f}$ uniformly on S.

(9) Prove that the product of uniformly convergent sequences of functions may not converge uniformly.

[Hint: Consider $f_n(x) = x + \frac{1}{n}$, $n \geq 1$ and $x \in \mathbb{R}$. Then consider f_n^2.]

(10) Prove that the sequence of functions defined by

$$f_n(x) = \frac{n+x}{4n+x}$$

converges uniformly on the interval $[0, r]$ for any $r > 0$, but does not converge uniformly on $[0, \infty)$.

(11) Let $\{f_n\}$ and $\{g_n\}$ be sequences of bounded functions on $S \subseteq \mathbb{R}$, and suppose $f_n \to f$ and $g_n \to g$ uniformly. Prove that $f_n g_n \to fg$ uniformly.

(12) Prove that $\{\sin(x + \frac{1}{n})\}$ converges uniformly to $\sin x$ on $\mathbb{R}$.

(13) Let f be uniformly continuous on $\mathbb{R}$ and let $\{r_n\}$ be a convergent sequence of real numbers. Suppose

$$f_n(x) = f(x + r_n) \qquad (n \geq 1, x \in \mathbb{R}).$$

Prove that $\{f_n\}$ converges uniformly on $\mathbb{R}$.