Indian Statistical Institute

Analysis II (HW - 6) Date: March 28, 2022 Instructor : Jaydeb Sarkar

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(1) Let $f \in C[0,1]$, and let

$$f_n(x) = f(x^n) \qquad (n \geq 1, x \in [0,1]).$$

Verify whether $\{\int_0^1 f_n\}_{n\geq 1}$ converges to $f(0)$.

(2) Prove that $\sum_{n=1}^{\infty} x^2 e^{-nx}$ converges uniformly on $(0,\infty)$.

(3) Prove that $\sum_{n=1}^{\infty} ne^{-nx}$ convergens uniformly on $[\epsilon,\infty)$ for any $\epsilon > 0$, but does not converge uniformly on $(0,\infty)$.

(4) Prove that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly on $(0,1)$.

(5) Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on $[-R,R]$ for all $R > 0$. Also, prove that

$$\frac{d}{dx}\Big(\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R}).$$

(6) Examine the uniform convergence of the following series:

$$(i) \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}\ , x \in \mathbb{R}, \quad (ii) \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\ , x \geq 0, \quad (iii) \sum_{n=1}^{\infty} \frac{\sin nx}{e^n}\ , x \in \mathbb{R}.$$

(7) Prove that the series

$$\sum_{n=0}^{\infty} \Big(\frac{x^{2n+1}}{2n+1} - \frac{x^{n+1}}{2n+2}\Big),$$

converges pointwise but not uniformly on $[0,1]$.

(8) Let $\sum f_n$ converges uniformly on $S \subseteq \mathbb{R}$. True or false?

  (i) $\{f_n\}$ is pointwise convergent on S.

  (ii) $\{f_n\}$ is uniformly convergent on S.

(9) Let $\{f_n\}$ converges uniformly on each $S_1, \ldots, S_m$. Prove that $\{f_n\}$ converges uniformly on $\cup_{k=1}^m S_k$.

(10) Give an example where $\{f_n\}$ converges uniformly on each of an infinite sequence of sets $S_1, S_2, \ldots$, but not on $\cup_{k=1}^{\infty} S_k$.