(1) Prove that if $f_n \to f$ pointwise on a finite set S, then the convergence is uniform.

(2) We say that $\sum f_n$ converges *absolutely uniformly* on S if $\sum |f_n|$ converges uniformly on S. Prove that if $\sum f_n$ converges absolutely uniformly on S, then $\sum f_n$ converges uniformly on S.

(3) Let $\sum a_n$ be a convergent series of real numbers. Prove that each of the following series is uniformly convergent on $[0, 1]$:

$$(i) \sum a_n \frac{x^n}{1 + x^n}. \qquad (ii) \sum a_n \frac{nx^n(1-x)}{1 + x^n}.$$

(4) Let $\{f_n\}$ be a sequence of functions on $[a, b]$ with the property that $f_n(x_n) \to 0$ for every convergent sequence $\{x_n\}$ in $[a, b]$. Prove that $\{f_n\}$ converges uniformly to zero on $[a, b]$.

(5) Let $\sum a_n$ be a convergent series of real numbers. Prove that $\sum a_n x^n$ is uniformly convergent on $[0, 1]$. Also prove that

$$\lim_{x \to 1-} \sum a_n x^n = \sum a_n.$$

(6) Use the Abel's test to prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} 2^{-nx},$$

converges uniformly on $[0, \infty)$.

(7) Let $0 < \epsilon < 2\pi$. Use the Dirichlet's test to prove that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}},$$

converges uniformly on $[\epsilon, 2\pi - \epsilon]$.

(8) Prove that $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ defines a continuous function on $(2n\pi, 2(n+1)\pi)$ for all $n \in \mathbb{Z}$.

(9) Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n},$$

on $[0, 1]$.

(10) Discuss the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n \sin nx}{n},$$

on $[0, 1]$.