Indian Statistical Institute

Analysis II (HW - 8) Date: April 3, 2022 Instructor : Jaydeb Sarkar

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(1) Prove that the series
$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2},$$
defines a continuous function on $\mathbb{R}$.

(2) Prove that
$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}\, dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}.$$

(3) Let $p > -1$. Prove that
$$\lim_{n\to\infty} \int_1^n (1 - \frac{x}{n})^n x^p\, dx = \int_1^{\infty} e^{-x} x^p\, dx.$$

(4) Can the series of functions $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ be differentiated term by term?

(5) Let $p > 0$. Consider the series $\sum_{n=1}^{\infty} \sin(\frac{x}{n^p})$. Prove that the series

(i) diverges for all $x \neq 0$ and $p \leq 1$, and

(ii) converges absolutely uniformly on bounded intervals, but not uniformly on $\mathbb{R}$ for all $p > 1$.

(6) Let f_n be continuous and $f_n \downarrow 0$ on $\mathbb{R}$. Prove that if $[a, b]$ does not contain any odd multiple of π, then $\sum_{n=1}^{\infty} (-1)^n f_n(x) \cos nx$ converges uniformly on $[a, b]$.

(7) Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ converges. Prove that $\sum_{n=1}^{\infty} \frac{1}{|x-a_n|}$ converges uniformly on bounded intervals not containing any a_n.

(8) Let $\{a_n\}$ be a monotonic sequence of real numbers such that $a_n \to 0$. Prove that the series
$$\sum_n a_n \sin nx \qquad \text{and} \sum_n a_n \cos nx,$$
converge uniformly on $\{x : \delta \leq |x| \leq \pi\}$, where $0 < \delta < \pi$.