Indian Statistical Institute

Analysis II (HW - 9) Date: April 8, 2022 Instructor : Jaydeb Sarkar

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(1) Suppose $m \in \mathbb{N}$, and let the power series $\sum a_n x^n$ has radius of convergence $R$. Prove that $\sum a_n x^{mn}$ has radius of convergence $R^{\frac{1}{m}}$.

(2) Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R_f, R_f)$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ on $(-R_g, R_g)$. Suppose $R = \min\{R_f, R_g\}$. Prove that

$$(f+g)(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n, \text{ and } f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n,$$

on $(-R, R)$, where

$$c_n = \sum_{j=0}^{n} a_{n-j} b_j \qquad (n \geq 0).$$

(3) Find a power series representation of $\frac{1}{x}$ centered at $r > 0$.

(4) Find a power series representation of $\log x$ centered at $r > 0$.

(5) Let $R_1$ and $R_2$ be the radii of convergence of $\sum_n a_n x^n$ and $\sum b_n x^n$, respectively. If

$$\limsup \left|\frac{a_n}{b_n}\right| < \infty,$$

then prove that $R_1 \geq R_2$.

(6) Find the radius of convergence, and the interval of convergence of the following power series:

$$(i) \sum_{n=0}^{\infty}(-1)^n x^{2^n}, \ (ii) \sum_{n=1}^{\infty} \frac{n!(x+3)^n}{n^n}, \ (iii) \sum_{n=1}^{\infty} \frac{2^n (x-3)^{n+1}}{n}, \ (iv) \sum_{n=2}^{\infty} \frac{(x+1)^n}{\log n}.$$

(7) Prove that

$$\sum_{n=0}^{\infty}(n+1)^2 x^n = \frac{1+x}{(1-x)^3} \qquad (|x| < 1).$$

(8) Find the radius of convergence of

$$\sum_{n=1}^{\infty}(1 + (-3)^{n-1})x^n.$$

(9) Prove that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

(10) Suppose the series of real numbers $\sum a_n$ converges conditionally. Prove that the radius of convergence of $\sum a_n x^n$ is 1.

(11) Prove that

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \cdot \qquad (|x| < 1).$$

(12) Suppose $f$ is analytic at $c$. Prove that $f'$ and $F$ are analytic at $c$, where $F(x) = \int_c^x f$.