

Riemann Integration.

Jaydeb Sarkar.

CALCULUS

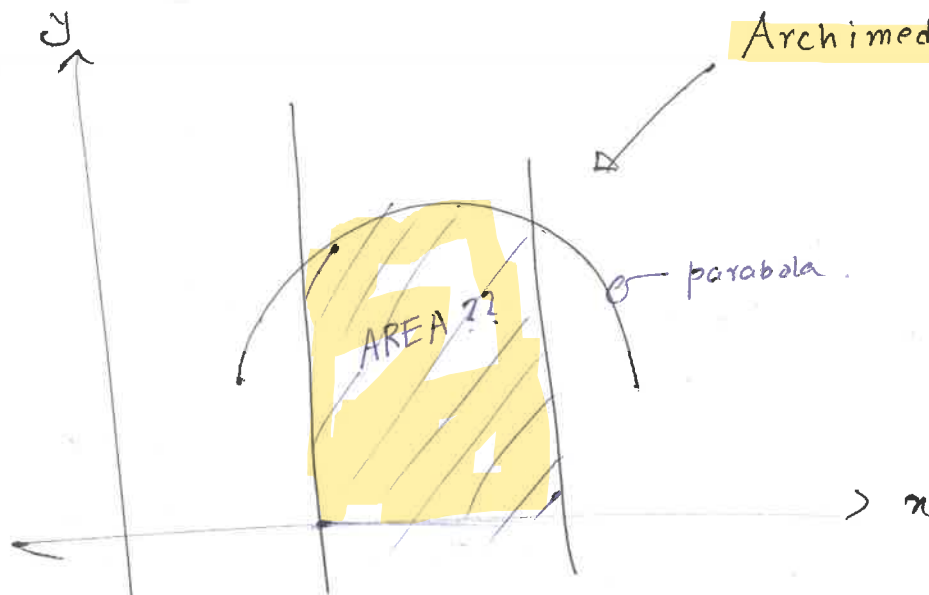
An old subject modified during 450 BC - till date.

"meaning": came from Latin - small stone / pebble.

Started by ("perhaps") : Antiphon (430 BC, Greece).

↓
Euclid (300 BC, Alexandria)

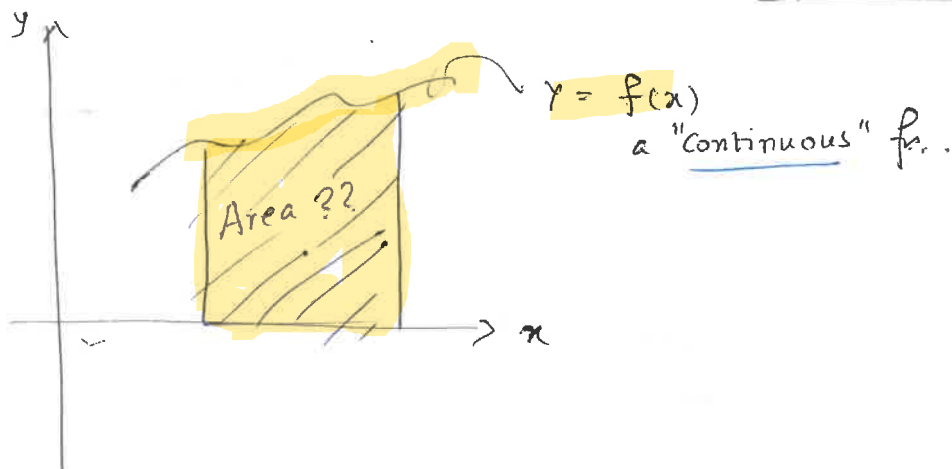
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Archimedes (250 BC, Greece).



Archimedes computed the area of a
parabolic segment (Also: area & circumference
of a circle).

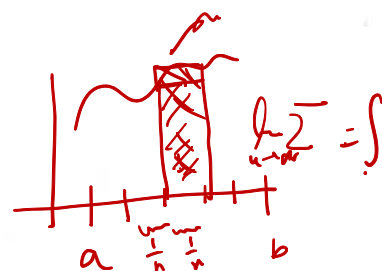
Then he asked:

Area of:



Ans: (Newton & ~~Leib~~ Leibniz : $\frac{1670}{11}$).

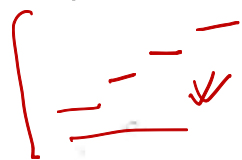
He needs it for his law of motion.



This integration is/was fine : AND that's our "School integration."

It is still the "BEST" BUT, Conceptually.

$\therefore$ ① School/Newton/Leibniz integration: $\{ f : [a, b] \rightarrow \mathbb{R} \text{ Cont. } f_n \}$.



② Riemann integration :

[He needed it for integrations of discnt. f_n (not too many discnt.) to study Fourier series.]
An integration which make sense for bdd & not "too" badly discnt. f_n .

What is the measurement of "too badly"?

③ Lebesgue integration : Deals with "highly" discnt. & unbdd f_n .

In master's : measure theory.

CALCULUS:

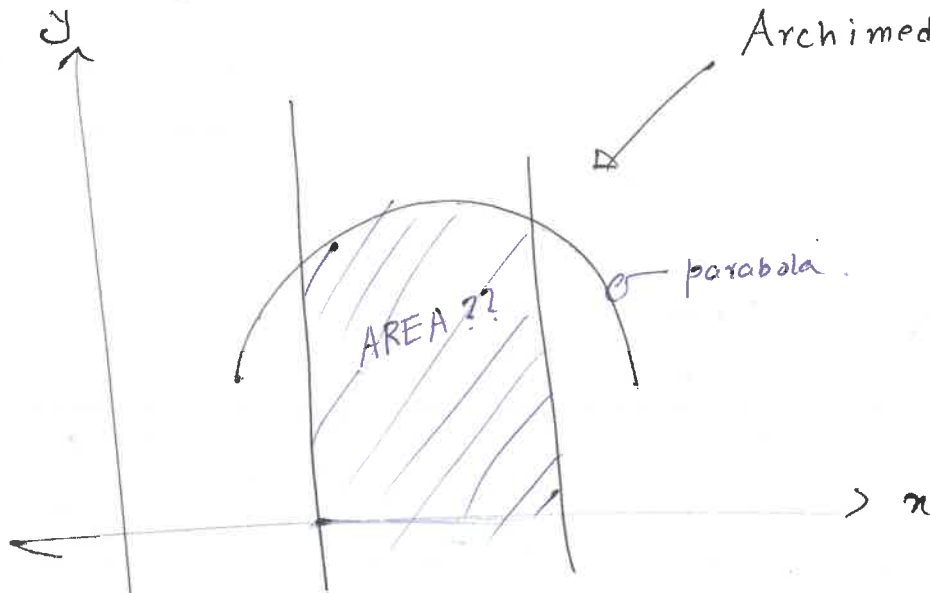
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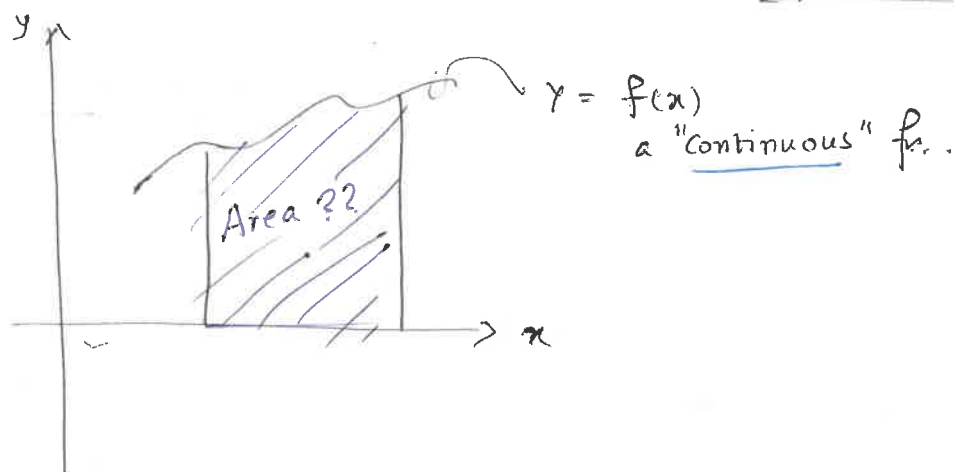
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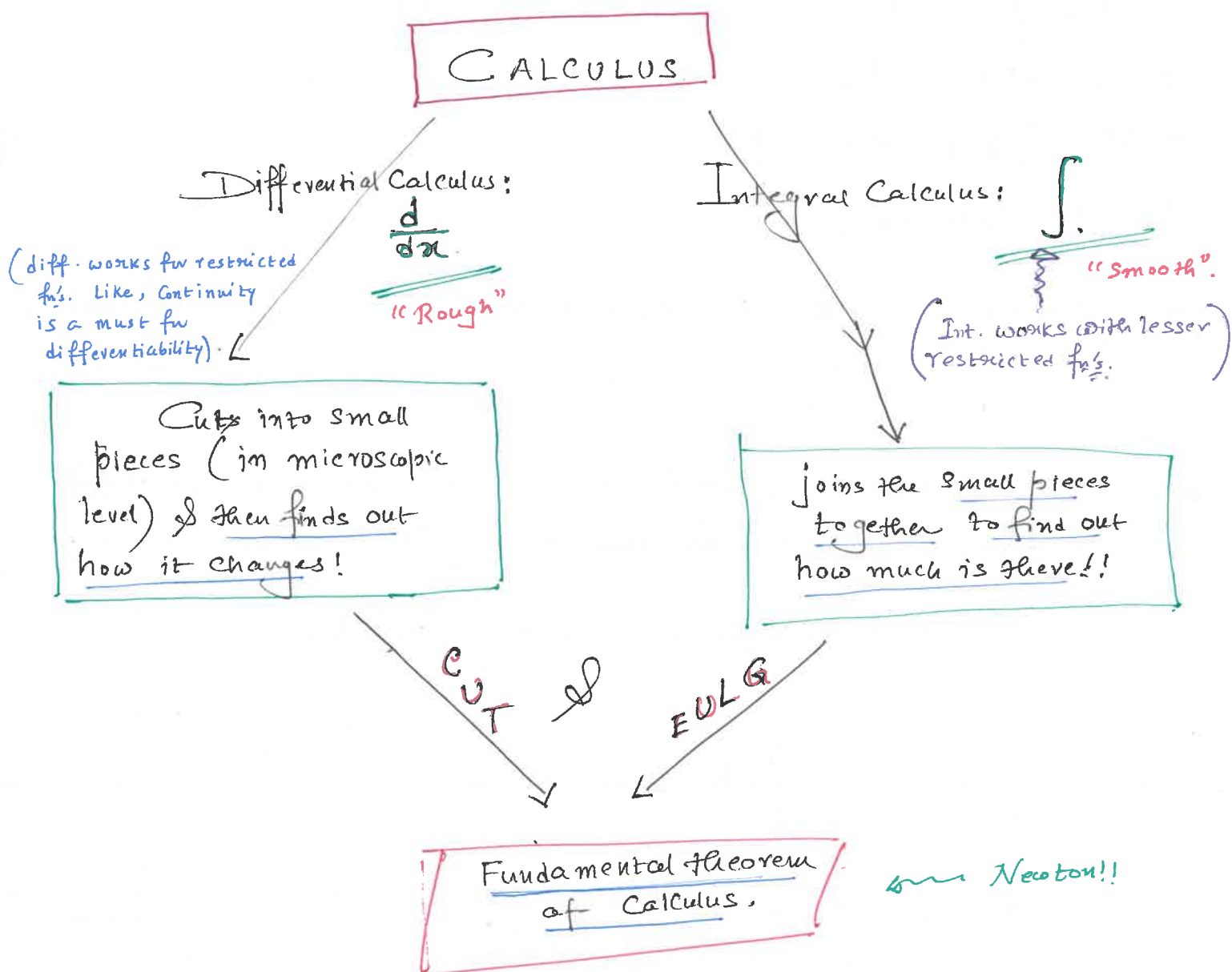
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② Riemann integration : An integration which make sense for bdd & not too "badly" disc. fn.
[He needed it for integrations of disc. fn (not too many disc.) to study Fourier series.]

Present interest

↓
③ Lebesgue integration : Deals with "highly" disc. & unbdd fns.

In master's measure theory.

Remember the following:



"Rough & Smooth": $\int$ makes fns "smoother".
 $\frac{d}{dx}$ makes fns "rougher".

Eg: Indefinite integration of a cont. fn exists & differentiable. Whereas, a diff. fn's derivative need not be ^{even} cont!!

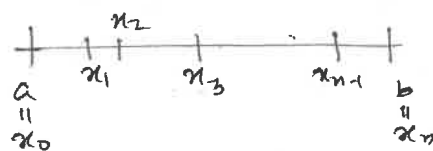
$\int$
 eg? — HW —

Let's start:

- # Always assume that $[a, b]$: a closed interval, $a < b$.
- # $\mathbb{N} = \{1, 2, \dots\}$.
- # $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.
- # $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.
- # $f: [a, b] \rightarrow \mathbb{R}$ is ALWAYS bounded (stated otherwise).

Def: A partition P of $[a, b]$ is a set of real n 's (called nodes) $\{x_0, x_1, \dots, x_n\}$, for some $n \in \mathbb{N}$ $\Rightarrow$.

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$



We ~~let~~ ^{denote by} I_j or P_j as $[x_{j-1}, x_j]$ is the j -th subinterval.
 $\forall j = 1, \dots, n.$

Def: If $I = (a, b)$ or $[a, b]$ or $[a, b)$ or $(a, b]$, then

$$|I| := b - a.$$

The length of I .

Fact: If $P: a = x_0 < x_1 < \dots < x_n = b$ is a partition of $I = [a, b]$, then

$$|I| = \sum_{j=1}^n |I_j|. \quad \text{--- HW ---}$$

Remark: Let P & $\tilde{P}$ be two partitions of $[a, b]$. Then $P \cup \tilde{P}$ is also a partition of $[a, b]$.

$$\left[P = \{x_0, x_1, \dots, x_n\}, \quad \tilde{P} = \{z_0, z_1, \dots, z_m\}, \text{ then } P \cup \tilde{P} = \{x_0, \dots, x_n, z_0, \dots, z_m\} \right. \\ \left. \begin{array}{c} \uparrow \\ \text{BUT } \underline{\text{ordered.}} \end{array} \right].$$

eg: $I = [0, 1]$. Then
$$0 < \underbrace{\frac{1}{8}}_{x_0} < \underbrace{\frac{1}{4}}_{x_1} < \underbrace{\frac{3}{4}}_{x_2} < \underbrace{1}_{x_4} \text{ is a partition of } I.$$

eg: $I = [0, 1]$. Then $P = \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \cup \{0\}$ is NOT a partition of P . — why?

Notation: 1) $\mathcal{P}[a, b]$ = $\mathcal{P}$ = the set of all partitions of $[a, b]$.

2) Suppose $f: [a, b] \rightarrow \mathbb{R}$ be a (BOUNDED) f_x .

Let $P \in \mathcal{P}$ with

$$P: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

$$\text{We set: } \underline{M_j} := \sup_{x \in I_j} f(x) \quad \& \quad \underline{m_j} := \inf_{x \in I_j} f(x),$$

$$\forall j = 1, \dots, n.$$

$$\text{Also, } \underline{M} := \sup_{x \in [a, b]} f(x) \quad \& \quad \underline{m} := \inf_{x \in [a, b]} f(x).$$

Observation: Let $S_1 \subseteq S_2 \subseteq \mathbb{R}$. Then

$\uparrow$
bdd.

$$\sup S_1 \leq \sup S_2 \quad \& \quad \inf S_1 \geq \inf S_2.$$

— HW —.

Corollary: $m \leq m_j \leq M_j \leq M \quad \forall j=1, \dots, n.$

Notation: $\mathcal{B}[a,b] = \text{Set of all bdd } f_n : [a,b] \rightarrow \mathbb{R}.$

Def: Let $f \in \mathcal{B}[a,b]$ & let $P: a = x_0 < x_1 < \dots < x_n = b$ be a partition of $[a,b]$. Then the upper Riemann sum of f w.r.t. P is defined by:

$$\underline{U(f; P)} := \sum_{j=1}^n M_j |I_j| \quad \left(= \sum_{j=1}^n \left(\sup_{x \in I_j} f(x) \right) \times (x_j - x_{j-1}) \right)$$

& the lower Riemann sum (w.r.t. P)

$$\underline{L(f; P)} = \sum_{j=1}^n m_j |I_j| \quad \left(= \sum_{j=1}^n \left(\inf_{x \in I_j} f(x) \right) \times (x_j - x_{j-1}) \right)$$

Note: Clearly, both $U(f; P)$ & $L(f; P)$ exist!!
[$\because f \in \mathcal{B}[a,b]$ & # nodes of $P < \infty$].

Thm: Given $f \in \mathcal{B}[a,b]$ ~~or $f \in \mathcal{B}[a,b]$~~ , we have

$$m(b-a) \leq L(f; P) \leq U(f; P) \leq M(b-a).$$

$$\forall P \in \mathcal{P}[a,b].$$

Proof:

Fix $P: a = x_0 < \dots < x_n = b$

Now $(b-a)$ $m \leq m_j \leq M_j \leq M \quad \forall j=1, \dots, n$

$$\Rightarrow m \cdot |I_j| \leq m_j |I_j| \leq M_j |I_j| \leq M |I_j|$$

$\forall j=1, \dots, n$

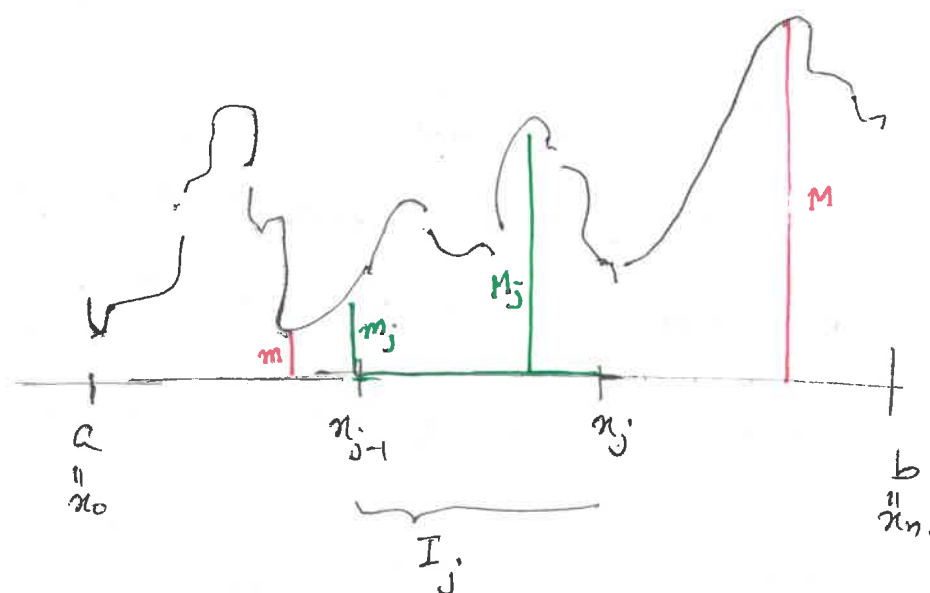
$[\because |I_j| > 0]$

$$\Rightarrow \underbrace{\sum_{j=1}^n m |I_j|}_{m(b-a)} \leq \sum_{j=1}^n m_j |I_j| \leq \sum_{j=1}^n M_j |I_j| \leq \underbrace{\sum_{j=1}^n M |I_j|}_{M(b-a)}$$

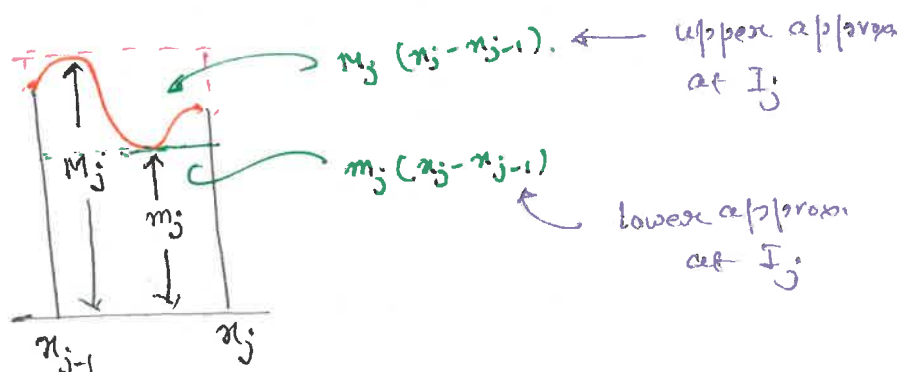
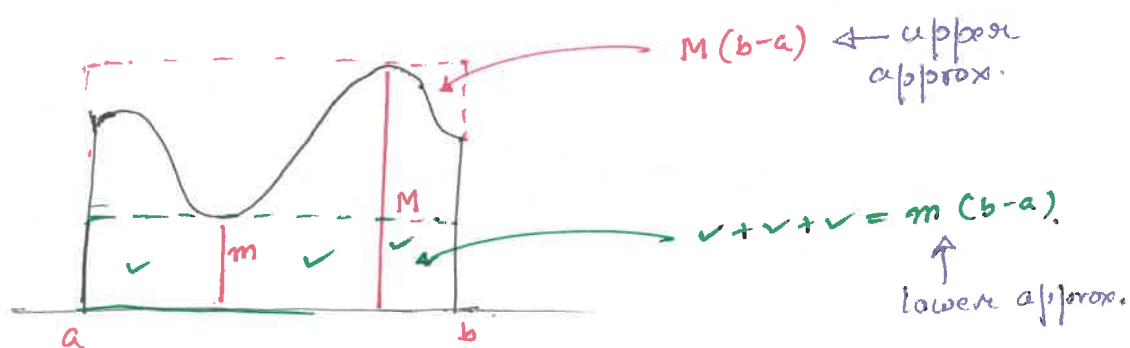
$$\Rightarrow m(b-a) \leq L(f; P) \leq U(f; P) \leq M(b-a).$$

□

The geometric implication of the above is pretty simple:
(OR interpretation)



More simple example:



And, Finally :

$$m(b-a) \leq L(f; P) \leq U(f; P) \leq M(b-a).$$

Independent
of P



Independent
of P .

$\forall P \in \mathcal{P}[a, b]$, both $L(f; P)$ & $U(f; P)$
are bdd by $m(b-a)$ & $M(b-a)$.

i.e. $L(f; P), U(f; P) \in [m(b-a), M(b-a)]$.

The next idea would be : more nodes $\Rightarrow$ increase $L(f; P)$,
decrease $U(f; P)$.

more $\uparrow$ & $\downarrow$ at a point,
 "If Possible".

In view of " $m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)$ ",
we are ready for the following ~~defin~~ definition: (*)

Def: Let $f \in \mathcal{B}[a, b]$. Define:

$$\int_a^b f = \sup \{ L(f, P) : P \in \mathcal{P}[a, b] \}.$$

Call it "The lower Riemann integration".

$$\int_a^b f = \inf \{ U(f, P) : P \in \mathcal{P}[a, b] \}.$$

Call it "The upper Riemann integration".

Remark: (*) $\Rightarrow \int_a^b f$ & $\overline{\int_a^b f}$ both exist !!

$\int_a^b f$ $\overline{\int_a^b f}$
 $\uparrow$ $\uparrow$
 " " " "
 $\int_a^b f$ $\int_a^b f$

Def: A fn $f \in \mathcal{B}[a, b]$ is said to be Riemann integrable, if

$$\int_a^b f = \overline{\int_a^b f} ;$$

In this case, we call the common value as
"the integration of f over $[a, b]$ " & write:

$$\int_a^b f = \int_a^b f = \overline{\int_a^b f}$$

Good! But how
to use?
We need tools !!

Wish list: We really want " $L(f, P) \leq U(f, Q)$, $\forall P \leq Q$ in $\mathcal{P}[a, b]$ " to be true!!
Indeed!!

Def: Let $P, \tilde{P} \in \mathcal{P}[a, b]$. We say that $\tilde{P}$ is a refinement of P [OR, $\tilde{P}$ is finer than P] if

$$x \in \tilde{P} \quad \forall x \in P.$$

$$\Updownarrow$$

$$\text{nodes of } \tilde{P} \supseteq \text{nodes of } P.$$

$\Updownarrow$
 i.e. $\tilde{P} = P \cup \hat{P}$, where $\hat{P}$ is a finite subset of $[a, b]$.

Notation: $\tilde{P} \supset P$ if $\tilde{P}$ is a refinement of P .

eg:



$$P = \{0, \frac{1}{3}, \frac{2}{3}, 1\} \subset \tilde{P} = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}.$$

Note: Let $P_1, P_2 \in \mathcal{P}[a, b]$. Set $\tilde{P} := P_1 \cup P_2$.

$$\text{Then } \tilde{P} \supset P_1 \quad \& \quad \tilde{P} \supset P_2.$$

Proof: Trivial!!

Proposition: Let $f \in \mathcal{B}[a, b]$ & $P, \tilde{P} \in \mathcal{P}[a, b]$. If $\tilde{P} \supset P$, then

$$L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, P).$$

↑
getting more ~~close~~ closer!!

Proof: " $L(f, \tilde{P}) \leq U(f, \tilde{P})$ " is known.

$\therefore$ Enough to prove " $L(f, P) \leq L(f, \tilde{P})$ " & " $U(f, \tilde{P}) \leq U(f, P)$ ".

We only prove the 1st one (as the 2nd one ~~will~~ be similar).

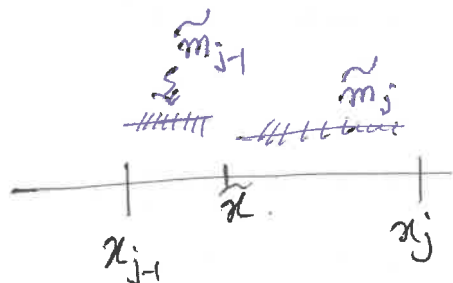
First, assume that $\tilde{P} := P \cup \{\tilde{x}\}$,

~~where~~ where $\tilde{x} \in [a, b] \setminus P$. [$\therefore \tilde{x}$ a new node.]

Set $P: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.

Then $\exists j \in \{1, \dots, n\}$ s.t.

$$x_{j-1} < \tilde{x} < x_j. \quad \leftarrow [\because x \in [a, b] \setminus \{a, b\}]$$

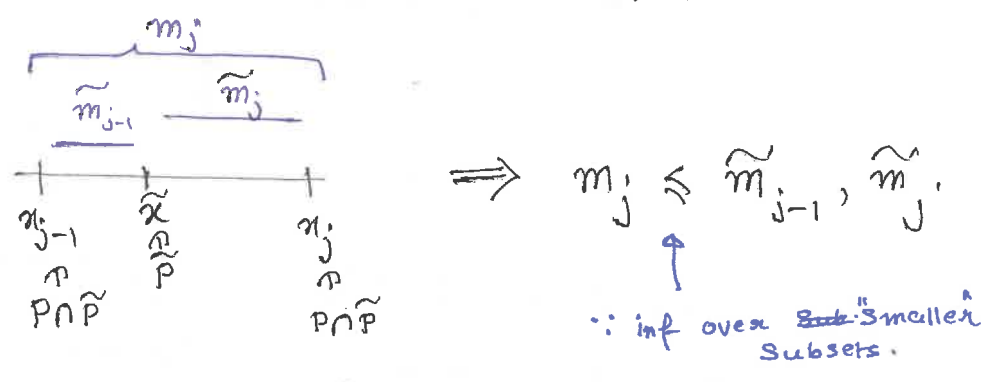


Set: $\tilde{m}_{j-1} := \inf \{f(x) : x \in [x_{j-1}, \tilde{x}]\}$

& $\tilde{m}_j := \inf \{f(x) : x \in [\tilde{x}, x_j]\}$.

$$\therefore L(f, \tilde{P}) - L(f, P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - x_{j-1})$$

Now



$$\begin{aligned} \Rightarrow L(f, \tilde{P}) - L(f, P) &= \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - x_{j-1}) \\ &= \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - x_{j-1}) \\ &= (\tilde{m}_{j-1} - m_j)(\tilde{x} - x_{j-1}) + (\tilde{m}_j - m_j)(x_j - \tilde{x}) \\ &\geq 0 \end{aligned}$$

$\Rightarrow L(f, \tilde{P}) \geq L(f, P)$. The general case: by induction.

The upper sum case: similar $\forall$ HW.

Cor: Let $f \in \mathcal{B}[a, b]$ & $P, Q \in \mathcal{P}[a, b]$. Then

$$L(f, P) \leq U(f, Q).$$

Proof: Let $\tilde{P} := P \cup Q \Rightarrow \tilde{P} \supset P, Q$.

$$\therefore L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, Q)$$

By applying the above prop. for $(\tilde{P}, P)$ & $(\tilde{P}, Q)$.

In particular: $L(f, P) \leq U(f, Q)$. $\square$

Cor: If $f \in \mathcal{B}[a, b]$, then

$$\underline{\int_a^b f} \leq \overline{\int_a^b f}.$$

Proof: We know: $L(f, P_1) \leq U(f, P_2) \quad \forall P_1, P_2 \in \mathcal{P}[a, b]$.

$\therefore$ For a fixed $P_2 \in \mathcal{P}[a, b]$,

$$\underline{\int_a^b f} = \sup_{P_1 \in \mathcal{P}[a, b]} L(f, P_1) \leq U(f, P_2).$$

$\therefore$ Taking inf on all over $P_2 \Rightarrow \underline{\int_a^b f} \leq \inf_{P_2} U(f, P_2) = \overline{\int_a^b f}.$

□

Notation: $\mathcal{R}[a, b] = \{f \in \mathcal{B}[a, b] : f \text{ is Riemann integrable}\}.$

Fact: Suppose $f \in \mathcal{B}[a, b]$. Then

$$f \in \mathcal{R}[a, b] \iff \underline{\int_a^b f} \geq \overline{\int_a^b f}.$$

Q: $\mathcal{B}[a, b] = \mathcal{R}[a, b] ?$

Ans: No!

Eg: Consider the Dirichlet f.s: $f: [0, 1] \rightarrow \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0, 1], \end{cases}$$

Clearly, $f \in \mathcal{B}[0, 1]$.

Suppose $P: 0 = x_0 < x_1 < \dots < x_n = 1$ be a partition of $[0, 1]$.

Recall: $I_j := [x_{j-1}, x_j]$.

$$\Rightarrow I_j \cap \mathbb{Q} \neq \emptyset \quad \& \quad I_j \cap \mathbb{Q}^c \neq \emptyset.$$

$$\forall j=1, \dots, n.$$

$$\Rightarrow m_j = 0 \quad \& \quad M_j = 1 \quad \forall j=1, \dots, n.$$

$$\therefore L(f, P) = 0 \quad \& \quad U(f, P) = 1. \quad [\text{By the def'n's. of } L \& U].$$

$$\forall P \in \mathcal{P}[0,1].$$

$$\Rightarrow \int_0^1 f = 0 \quad \neq \quad 1 = \int_0^1 f.$$

$$\Rightarrow f \notin \mathcal{R}[0,1].$$

□

eg: ($\mathcal{R}[a,b] \neq \emptyset$):

Fix $c \in \mathbb{R}$ & define $f(x) = c \quad \forall x \in [a,b]$.

$$\text{Then, } \forall P \in \mathcal{P}[a,b], \quad L(f, P) = c \times (b-a) = U(f, P).$$

↑ Why? check.

$$\Rightarrow \int_a^b f = c \times (b-a) = \int_a^b f$$

$$\Rightarrow f \in \mathcal{R}[a,b] \quad \& \quad \int_a^b f = c(b-a).$$

□

eg: $\exists f$ s.t. $|f| \in \mathcal{R}[a,b]$ but $f \notin \mathcal{R}[a,b]$.

$$\text{Consider } f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}.$$

Clearly, $f \in \mathcal{B}[0,1]$. Here $|f| \equiv 1 \Rightarrow |f| \in \mathcal{R}[0,1]$.

But $f \notin \mathcal{R}[0,1]$, $\leftarrow$ HW.

□

Proposition: Let $f \in \mathcal{B}[a, b]$ & $P, \tilde{P} \in \mathcal{P}[a, b]$. If $\tilde{P} \supset P$, then

$$\underbrace{L(f, P) \leq L(f, \tilde{P})}_{(1) \checkmark} \leq \underbrace{U(f, \tilde{P}) \leq U(f, P)}_{(2) \checkmark}$$

↑
known.
getting more closer to closer!!

Proof: " $L(f, \tilde{P}) \leq U(f, \tilde{P})$ " is known.

$\therefore$ Enough to prove " $L(f, P) \leq L(f, \tilde{P})$ " & " $U(f, \tilde{P}) \leq U(f, P)$ ".

We only prove the 1st one (as the 2nd one will be similar).

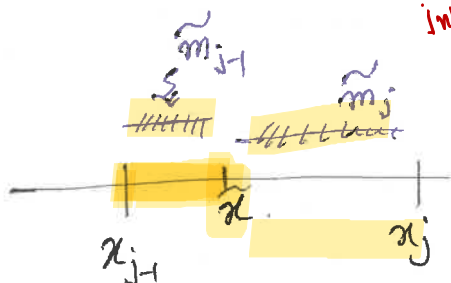
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where $\tilde{x} \in [a, b] \setminus P$. [$\therefore \tilde{x}$ a new node.]

Set $P: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.

Then $\exists j \in \{1, \dots, n\}$ s.t.

$$x_{j-1} < \tilde{x} < x_j$$



$$\checkmark \tilde{P}: a = x_0 < x_1 < \dots < x_{j-1} < \tilde{x} < x_j < \dots < x_n = b$$

Set: $\tilde{m}_{j-1} := \inf \{f(x) : x \in [x_{j-1}, \tilde{x}]\}$

& $\tilde{m}_j := \inf \{f(x) : x \in [\tilde{x}, x_j]\}$

$$m_1 |I_1| + \dots + m_{j-1} |I_{j-1}| + m_j |I_j| + m_{j+1} |I_{j+1}| + \dots + m_n |I_n|$$

$$\therefore L(f, \tilde{P}) - L(f, P) = \tilde{m}_{j-1} (\tilde{x} - x_{j-1}) + \tilde{m}_j (x_j - \tilde{x})$$

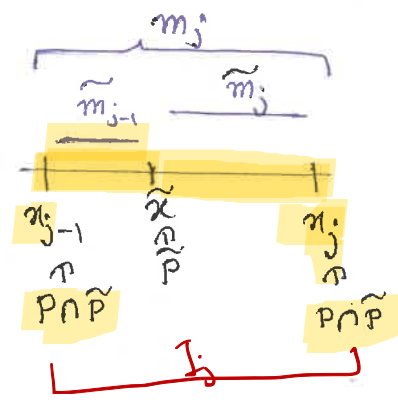
$$- m_j (x_j - x_{j-1})$$

$$I_j = [x_{j-1}, \tilde{x}]$$

$$m_j = \inf_{I_j} f(x) \leq \inf_{[x_{j-1}, \tilde{x}]} f(x)$$

~~f(x)~~

Now



$$\Rightarrow m_j \leq \tilde{m}_{j-1}, \tilde{m}_j$$

$\therefore$ inf over smaller subsets.

$$\Rightarrow L(f, \tilde{P}) - L(f, P) = \tilde{m}_{j-1} (\tilde{x} - x_{j-1}) + \tilde{m}_j (x_j - \tilde{x}) - m_j (x_j - x_{j-1})$$

$$[\because x_j - x_{j-1} = (x_j - \tilde{x}) + (\tilde{x} - x_{j-1})]$$

$$= (\tilde{m}_{j-1} - m_j) (\tilde{x} - x_{j-1}) + (\tilde{m}_j - m_j) (x_j - \tilde{x})$$

$$\geq 0$$

$$\Rightarrow L(f, \tilde{P}) \geq L(f, P)$$

The general case: by induction.

The upper sum case: Similar & HW.

Cor: Let $f \in \mathcal{B}[a, b]$ & $P, Q \in \mathcal{P}[a, b]$. Then

$$L(f, P) \leq U(f, Q)$$

Proof: Let $\tilde{P} := P \cup Q \Rightarrow \tilde{P} \supset P, Q$

By applying the above prop. for $(\tilde{P}, P)$ & $(\tilde{P}, Q)$.

$$\therefore L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, Q)$$

where $X = P \cup Q$.

$$L(f, P) \leq U(f, \tilde{P}) \leq U(f, Q)$$

$$\text{In particular: } L(f, P) \leq U(f, Q)$$

Cor: If $f \in \mathcal{B}[a, b]$, then

$$\int_a^b f \leq \overline{\int_a^b f}.$$

Proof: We know: $L(f, P_1) \leq U(f, P_2) \quad \forall P_1, P_2 \in \mathcal{P}[a, b]$.

$\therefore$ For a fixed $P_2 \in \mathcal{P}[a, b]$,

$$\int_a^b f = \sup_{P_1 \in \mathcal{P}[a, b]} L(f, P_1) \leq U(f, P_2).$$

$\therefore$ Taking inf on all over $P_2 \Rightarrow \int_a^b f \leq \inf_{P_2} U(f, P_2) = \overline{\int_a^b f}.$

□

Notation: $\mathcal{R}[a, b] = \{ f \in \mathcal{B}[a, b] : f \text{ is Riemann integrable} \}.$

Fact: Suppose $f \in \mathcal{B}[a, b]$. Then

$$f \in \mathcal{R}[a, b] \iff \int_a^b f \geq \overline{\int_a^b f}.$$

Q: $\mathcal{B}[a, b] = \mathcal{R}[a, b]$?

Ans: No!

eg: Consider the Dirichlet $f_D: [0, 1] \rightarrow \mathbb{R}$ defined by:

$$f_D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0, 1] \end{cases}$$

Clearly, $f_D \in \mathcal{B}[0, 1]$.

Suppose $P: 0 = x_0 < x_1 < \dots < x_n = 1$ be a partition of $[0, 1]$.

$f_D \notin \mathcal{R}[a, b]$
 $\Rightarrow \int f_D \neq \overline{\int f_D}$
 $\Rightarrow \int f_D < \overline{\int f_D}$

Recall: $I_j := [x_{j-1}, x_j]$.

$$\Rightarrow I_j \cap \mathbb{Q} \neq \emptyset \text{ \& } I_j \cap \mathbb{Q}^c \neq \emptyset, \quad \forall j=1, \dots, n.$$

$$\Rightarrow m_j = 0 \text{ \& } M_j = 1 \quad \forall j=1, \dots, n.$$

$$\therefore L(f, P) = 0 \text{ \& } U(f, P) = 1. \quad [\text{By the def'n's. of } L \text{ \& } U].$$

$$[m_j | I_j] = 0.$$

$$[M_j | I_j] = 1.$$

$$\forall P \in \mathcal{P}[0,1], \quad L(f, P) = 0, \quad U(f, P) = 1.$$

$$\Rightarrow \int_0^1 f = 0 \neq 1 = \int_0^1 f.$$

$$\Rightarrow f \notin R[0,1].$$

$$\therefore \mathbb{Q}[a,b] \neq R[a,b]. \quad \square$$

$$f \in R[0,1] \text{ if } \int f = \int f. \quad (\Rightarrow)$$

$$\int f = \sup_{P \in \mathcal{P}[0,1]} L(f, P) = 0$$

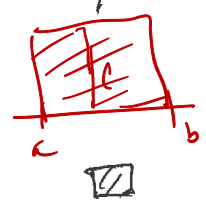
eg: ($R[a,b] \neq \emptyset$):

Fix $c \in \mathbb{R}$ \& define $f(x) = c \quad \forall x \in [a,b]$.

$$\text{Then, } \forall P \in \mathcal{P}[a,b], \quad L(f, P) = c \times (b-a) = U(f, P).$$

$$\text{If } P \in \mathcal{P}[a,b], \quad M_j = c, \quad m_j = c \Rightarrow \int f = c(b-a)$$

Why? check.



$$\Rightarrow \int_a^b f = c \times (b-a) = \int_a^b f$$

$$\Rightarrow f \in R[a,b] \text{ \& } \int_a^b f = c(b-a).$$

eg: $\exists f$ s.t. $|f| \in R[a,b]$ but $f \notin R[a,b]$.

$$\text{Consider } f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Clearly, $|f| \in R[0,1]$. Here $|f| \equiv 1 \Rightarrow |f| \in R[0,1]$.
But $f \notin R[0,1]$, $\leftarrow$ HW. $\square$

$$\int f = -1 \neq 1 = \int f \Rightarrow f \notin R[0,1].$$

Note

In view of the above example (i.e. $f \notin \mathcal{R}[a, b]$ but $|f| \in \mathcal{R}[a, b]$), we have the following question:

$$f \in \mathcal{R}[a, b] \stackrel{?}{\implies} |f| \in \mathcal{R}[a, b] \quad ?$$

— Think about it —

" $\mathcal{R}[a, b]$ is invariant under $| \cdot |$ "?

Goal

Thm: Let $f \in \mathcal{B}[a, b]$. Then $f \in \mathcal{R}[a, b] \iff \forall \epsilon > 0 \exists P \in \mathcal{P}[a, b]$ s.t. $\underbrace{U(f, P) - L(f, P)}_{\geq 0} < \epsilon$.

Fixed $f \in \mathcal{B}[a, b]$,
define $\eta_f: \mathcal{P}[a, b] \rightarrow \mathbb{R}$
 $\eta_f(P) = U(f, P) - L(f, P)$
 $\eta_f(P) < \epsilon$
for some $P \in \mathcal{P}[a, b]$

Proof: " $\Leftarrow$ " Let $\epsilon > 0$.

$$\therefore \exists P \in \mathcal{P}[a, b] \text{ s.t. } U(f, P) - L(f, P) < \epsilon$$

Now $L(f, P) \leq \int_a^b f \leq U(f, P)$ (true in general)

$$\Rightarrow \int_a^b f \leq U(f, P) < \epsilon + L(f, P)$$

(By assumption)

But $L(f, P) \leq \int_a^b f$

$$\therefore \int_a^b f < \epsilon + \int_a^b f$$

$$\Rightarrow \int_a^b f - \int_a^b f < \epsilon$$

We also know, in general, that $\int_a^b f \leq \int_a^b f$.

$$\therefore 0 \leq \int_a^b f - \int_a^b f < \epsilon, \quad \forall \epsilon > 0.$$
$$\Rightarrow \int_a^b f = \int_a^b f \Rightarrow f \in \mathcal{R}[a, b]$$

$\Rightarrow$ Suppose $f \in \mathcal{R}[a, b]$. Let $\varepsilon > 0$.

Handwritten notes:
 $\inf_P U(f, P) \Rightarrow \int f = \int_a^b f = \int_a^b f$
 $\sup_P L(f, P)$
 (16) $\sup A = m$
 $\forall \varepsilon > 0, \exists x \in A$ s.t.
 $x > m - \varepsilon$

$$\therefore \exists P_1 \in \mathcal{P}[a, b] \text{ s.t.}$$

$$L(f, P_1) > \int_a^b f - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2} \quad (A)$$

$$\& \exists P_2 \in \mathcal{P}[a, b] \text{ s.t.}$$

$$U(f, P_2) < \int_a^b f + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2} \quad (B)$$

Set $P := P_1 \cup P_2$. $\Rightarrow P \supset P_1 \& P_2$.

Handwritten note: Common refinement

Claim: $U(f, P) - L(f, P) < \varepsilon$.

Now, $U(f, P) \leq U(f, P_2) < \int_a^b f + \frac{\varepsilon}{2}$.
 $(\because P \supset P_2)$ (B)

$$< L(f, P_1) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

(A)

$$< L(f, P) + \varepsilon$$

($\because P \supset P_1$)

$$\Rightarrow U(f, P) - L(f, P) < \varepsilon.$$

□

Note: We always have the following:

$$0 \leq U(f, P) - L(f, P) \quad \forall P$$

Def: Let $P \in \mathcal{P}[a, b]$. Then the mesh of P (or norm of P) is defined as:

$$\|P\| := \max \{x_j - x_{j-1} : 1 \leq j \leq n\},$$

where $P: a = x_0 < x_1 < \dots < x_n = b$.

A kind of continuity property of Riemann integ.

Thm: (Darboux thm) Let $f \in \mathcal{B}[a, b]$. Then $f \in \mathcal{R}[a, b] \iff$
for $\epsilon > 0 \exists \delta > 0$ s.t.

$$U(f, P) - L(f, P) < \epsilon$$

(for some $P \in \mathcal{P}[a, b]$)

$$\forall P \in \mathcal{P}[a, b] \text{ with } \|P\| < \delta.$$

[For $f \in \mathcal{B}[a, b]$ fixed, define $\eta : \mathcal{P}[a, b] \rightarrow \mathbb{R}_{\geq 0}$ by
 $\eta(P) = U(f, P) - L(f, P) \quad \forall P \in \mathcal{P}[a, b]$.

So, $f \in \mathcal{R}[a, b] \iff$ for $\epsilon > 0 \exists \delta > 0$ s.t.

$$\eta(P) = |U(f, P) - L(f, P)| < \epsilon \quad \forall \|P\| < \delta !!$$

Proof: " $\Leftarrow$ " : Follows from ~~prev~~ the last observation.

" $\Rightarrow$ " Let $f \in \mathcal{R}[a, b]$ & let $\epsilon > 0$.

$$\text{Goal: } \exists P \in \mathcal{P}[a, b] \text{ s.t. } U(f, P) - L(f, P) < \epsilon.$$

[Again, by the prev. observation, we will conclude that

$$\therefore \exists \tilde{P} \in \mathcal{P}[a, b] \text{ s.t. } U(f, \tilde{P}) - L(f, \tilde{P}) < \frac{\epsilon}{2}.$$

(By the prev. obs.) (*)

Assume that $\# \text{ nodes of } \tilde{P} = p.$

" $\therefore f \in \mathcal{B}[a, b]$, $\exists M > 0$ s.t.

$$|f(x)| \leq M \quad \forall x \in [a, b]$$

Our friendly bound.

Set $\delta := \frac{\epsilon}{8pM}.$

Suggested by a back calculation.

$(\mathcal{P}[a, b], \|\cdot\|) \rightarrow \mathbb{R}$

$f: [a, b] \rightarrow \mathbb{R}$ is cont. at c
for $\epsilon > 0 \exists \delta > 0$ s.t.
 $|f(x) - f(c)| < \epsilon$
 $\forall |x - c| < \delta$

$(\mathbb{R}_{\geq 0}, |\cdot|)$
 $\uparrow \eta$
 $(\mathcal{P}[a, b], \|\cdot\|)$

Let $P \in \mathcal{P}[a,b]$ & suppose $\|P\| \leq \delta$. (fix P)
 $\underbrace{\|P\|}_{\text{max length of subintervals.}}$

Set $\hat{P} := P \cup \tilde{P} \leftarrow \therefore \hat{P} \supset P, \tilde{P}$.

$\Rightarrow \hat{P}$ has at most p nodes that are not in P . ~~***~~

Now, let $\hat{P} = P \cup \{\tilde{x}\}$ & $\tilde{x} \notin P$. [i.e. $p=1$ case].

As earlier: Set $P: a = x_0 < x_1 < \dots < x_m = b$.
 $\uparrow$
(p-12) & assume $x_{j-1} < \tilde{x} < x_j$.

Then:

$$L(f, \hat{P}) - L(f, P) = (\tilde{m}_{j-1} - m_j)(\tilde{x} - x_{j-1}) + (\tilde{m}_j - m_j)(x_j - \tilde{x}).$$

[See page - 12]

$$\leq 2M \|P\|.$$

By, if $\hat{P} = P \cup \{\tilde{x}_1, \dots, \tilde{x}_p\}$, then (by induction),

$$L(f, \hat{P}) - L(f, P) \leq 2Mp \|P\| < 2Mp \times \delta = \frac{\epsilon}{4}.$$

$\therefore L(f, \hat{P}) - L(f, P) < \frac{\epsilon}{4}$ } $\text{--- } \textcircled{+}$
By $U(f, P) - U(f, \hat{P}) < \frac{\epsilon}{4}$ } ~~***~~

$\therefore \textcircled{+} \Rightarrow U(f, P) - L(f, P) < \frac{\epsilon}{2} + (U(f, \hat{P}) - L(f, \hat{P}))$ ~~***~~

But $\textcircled{*} \Rightarrow U(f, \tilde{P}) - L(f, \tilde{P}) < \frac{\epsilon}{2}$. $\therefore \hat{P} \supset \tilde{P}$, we know.

$L(f, \tilde{P}) \leq L(f, \hat{P})$ & $U(f, \tilde{P}) \geq U(f, \hat{P})$.

$\Rightarrow U(f, \hat{P}) - L(f, \hat{P}) \leq U(f, \tilde{P}) - L(f, \tilde{P}) < \epsilon/2$. ~~***~~

$$\therefore (**) \Rightarrow U(f, P) - L(f, P) < \varepsilon.$$

□

Notation: $C[a, b] = \{ f: [a, b] \rightarrow \mathbb{R} \text{ continuous} \}$

roots of f in $[a, b]$

$\{ \text{polynomials} \} \subseteq C[a, b]$. AND $\{ \text{rationals} \} \subseteq C[a, b]$.

Also: $\{ e^x, \sin x, \cos x, \dots \} \subseteq C[a, b]$.

$\frac{1}{2} \in \mathbb{Q}$
 $\mathbb{Q} \cap [a, b] = \emptyset$
 $\frac{1}{2} \in C[a, b]$

Who are they??

Any relation $\{ e^x, \sin x, \cos x, \dots \} \leftrightarrow \mathbb{R}[x]$??

! This is a large class!!

A pending question.

Thm: $C[a, b] \subseteq R[a, b]$.

Proof: Let $f \in C[a, b]$.

$\Rightarrow f: [a, b] \rightarrow \mathbb{R}$ is uniformly continuous.

Let $\varepsilon > 0$. By uniform cont. $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \forall x, y \in [a, b] \text{ s.t. } |x - y| < \delta.$$

The new " ε ".

Let $P \in \mathcal{P}[a, b]$ & assume $\|P\| < \delta$. Fix P .

Set $P: a = x_0 < x_1 < \dots < x_n = b$.

Now $f|_{[x_{j-1}, x_j]}: [x_{j-1}, x_j] \rightarrow \mathbb{R}$ is also unif. cont. $\forall j = 1, \dots, n$.

$\Rightarrow f|_{[x_{j-1}, x_j]}$ assumes its max (which is M_j) & min (which is m_j) in $[x_{j-1}, x_j]$ $\forall j = 1, \dots, n$.

$\therefore \|P\| < \delta$, we know

$$\eta_j - \eta_{j-1} < \delta \quad \forall j=1, \dots, n.$$

The max length
of subintervals

In particular: $|x-y| < \delta \quad \forall x, y \in [\eta_{j-1}, \eta_j] \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

$$\therefore M_j - m_j < \frac{\varepsilon}{b-a} \quad \forall j=1, \dots, n$$

$$\therefore U(f, P) - L(f, P) = \sum_{j=1}^n M_j (\eta_j - \eta_{j-1}) - \sum_{j=1}^n m_j (\eta_j - \eta_{j-1})$$

$$= \sum_{j=1}^n (M_j - m_j) (\eta_j - \eta_{j-1})$$

$$< \frac{\varepsilon}{b-a} \times \sum_{j=1}^n (\eta_j - \eta_{j-1})$$

$$= \frac{\varepsilon}{b-a} \times b-a$$

$$= \varepsilon.$$

$$\Rightarrow U(f, P) - L(f, P) < \varepsilon.$$

$$\Rightarrow f \in R[a, b].$$

□

Q: Still, how to compute

$$\int_a^b f \quad \text{for } f \in C[a, b]$$

??

- WAIT -

On how to compute $\int_a^b f$? where $f \in \mathcal{R}[a, b]$

1st attempt: (Sequential approach)

Thm: Let $f \in \mathcal{B}[a, b]$. Then $f \in \mathcal{R}[a, b] \iff \exists$ a seqn. $\{P_n\} \subseteq \mathcal{P}[a, b]$

s.t.
$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, in this case: $\{r_n\} \subseteq \mathbb{R}_{\geq 0}$ # $P_n \hookrightarrow \infty$

$$\int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n).$$

Proof: [Note: $\because U(f, P) - L(f, P) \geq 0 \forall P \in \mathcal{P}[a, b]$,

$\tilde{P} \geq P$
 $U(f, \tilde{P}) - L(f, \tilde{P}) \leq U(f, P) - L(f, P)$
 $\{U(f, P_n) - L(f, P_n)\} \subseteq \mathbb{R}_{\geq 0} \quad (*)$

" $\Rightarrow$ " Let $f \in \mathcal{R}[a, b]$. We know: "for $\varepsilon > 0$, $\exists P_\varepsilon \in \mathcal{P}[a, b]$
 $\Rightarrow U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$ ".

$\therefore$ For each $n \in \mathbb{N}$, $\exists P_n \in \mathcal{P}[a, b]$ s.t. Call it "Cauchy Criterion"

$$U(f, P_n) - L(f, P_n) < \frac{1}{n}.$$

By $(*)$ above: $U(f, P_n) - L(f, P_n) \rightarrow 0$
as $n \rightarrow \infty$.

" $\Leftarrow$ " If $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$, then for $\varepsilon > 0$

$\exists N_0 \in \mathbb{N}$ s.t. $U(f, P_n) - L(f, P_n) < \varepsilon$ # $n \geq N_0$
 $\Rightarrow f \in \mathcal{R}[a, b]$. By Cauchy Criterion.

Finally, if $f \in \mathcal{R}[a, b]$, then for $\{P_n\} \subseteq \mathcal{P}[a, b]$
as above:

We have:

$$0 \leq U(f, P_n) - \overline{\int_a^b f} \quad \leftarrow (\because \overline{\int f} = \inf U(f, P))$$

$$= U(f, P_n) - \int_a^b f \quad \leftarrow (\because f \in \mathcal{R}[a, b])$$

(As $\int f = \sup_P L(f, P)$) $\Rightarrow U(f, P_n) - L(f, P_n) \xrightarrow{\text{as } n \rightarrow \infty} 0$

$$\Rightarrow \lim_{n \rightarrow \infty} U(f, P_n) = \int_a^b f = \int_a^b f.$$

Finally, $L(f, P_n) = U(f, P_n) - (U(f, P_n) - L(f, P_n))$

$$\int_a^b f - 0 \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow \lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f. \quad \square$$

DANGER:

$a_n - b_n \rightarrow 0 \not\Rightarrow$ Even $\lim a_n$ &/or $\lim b_n$ exist !!

Remark:

Evidently, if $\exists \{P_n\} \subseteq \mathcal{P}[a, b]$ s.t.

$$L(f, P_n) \rightarrow c \text{ \& } U(f, P_n) \rightarrow c, \text{ then}$$

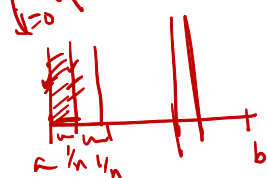
$$f \in \mathcal{R}[a, b] \text{ \& } \int_a^b f = c.$$

A nice way to prove existence of $\mathcal{R}$ -integrability

(evaluating too) of bdd fns.

[Reminding us
Newton].

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f(a + \frac{j}{n}) \times \frac{1}{n}$$



$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f(a + \frac{j}{n}) \times \frac{1}{n} = \int_a^b f$$

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n f(a + \frac{j}{n}) \times \frac{1}{n} = \int_a^b f$$



Eg: $f(x) = x^2$ $x \in [0, 1]$.

Fix $n \in \mathbb{N}$.

Consider "the classical" partition:

Why call it classical?

$P_n: 0 = x_0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \dots < x_{n-1} = \frac{n-1}{n} < x_n = 1$.

$\therefore I_j = \left[\frac{j-1}{n}, \frac{j}{n} \right] \quad \forall j = 1, \dots, n$.

$\therefore f$ is $\uparrow$, it follows that:

$m_j = \left(\frac{j-1}{n} \right)^2 \quad \& \quad M_j = \left(\frac{j}{n} \right)^2$.

$\forall j = 1, \dots, n$.

$\therefore U(f, P_n) = \sum_{j=1}^n M_j \times \frac{1}{n} = \sum_{j=1}^n \frac{j^2}{n^2} \times \frac{1}{n}$.

$= \frac{1}{n^3} \sum_{j=1}^n j^2$

$= \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2) = \frac{1}{n^3} \times \frac{1}{6} \times n \times (n+1)(2n+1)$

$= \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right)$

Also, $L(f, P_n) = \sum_{j=1}^n m_j \frac{1}{n} = \frac{1}{n^3} \sum_{j=1}^n (j-1)^2$

$= \frac{1}{n^3} (1^2 + 2^2 + \dots + (n-1)^2)$

$= \frac{1}{n^3} \times \frac{1}{6} \times (n-1) \times n \times (2(n-1)+1)$

$= \frac{1}{6} \times \left(1 - \frac{1}{n} \right) \times \left(2 - \frac{1}{n} \right)$.

$\therefore U(f, P_n) \rightarrow \frac{1}{3} \quad \& \quad L(f, P_n) \rightarrow \frac{1}{3}$.

$$\Rightarrow U(f, P_n) - L(f, P_n) \longrightarrow 0.$$

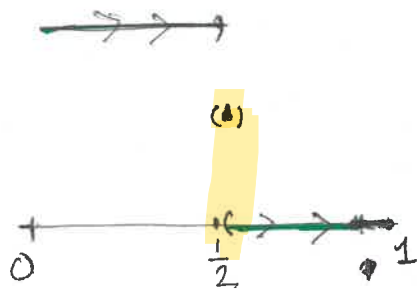
$$\Rightarrow \underline{f \in R[0,1]} \quad \& \quad \underline{\int_0^1 f = \lim_{n \rightarrow \infty} U(f, P_n) = \frac{1}{3}}.$$

Q: $R[a,b] \setminus C[a,b] \neq \varnothing$?

i.e. $\exists f \in R[a,b]$ s.t. $f \in R[a,b]$ but $f \notin C[a,b]$??

Ans: Yes.

Eg:



$$f(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & \frac{1}{2} < x \leq 1. \end{cases}$$

Claim: $f \in R[0,1]$, Clearly, $f \notin C[0,1]$.

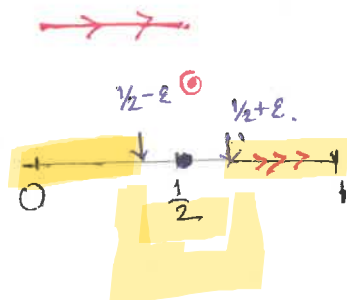
Proof

Let $\varepsilon > 0$ (small).

Consider the partition: $P_\varepsilon: 0 = x_0 < x_1 = \frac{1}{2} - \varepsilon < x_2 = \frac{1}{2} + \varepsilon < x_3 = 1$

$$\text{i.e. } 0 < \frac{1}{2} - \varepsilon < \frac{1}{2} + \varepsilon < 1.$$

A 2-node partition.



$$\therefore \left. \begin{aligned} I_1 &= [0, \frac{1}{2} - \varepsilon] \\ I_2 &= [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \\ I_3 &= [\frac{1}{2} + \varepsilon, 1] \end{aligned} \right\}$$

$$\therefore \left. \begin{aligned} m_1 &= 1, & m_2 &= 0, & m_3 &= 0 \\ M_1 &= 1, & M_2 &= 1, & M_3 &= 0 \end{aligned} \right\}$$

$$\begin{aligned}\therefore \underline{L(f, P_\varepsilon)} &= \sum_{j=1}^3 m_j |I_j| \\ &= 1 \times \left(\frac{1}{2} - \varepsilon\right) + 0 + 0 \\ &= \frac{1}{2} - \varepsilon.\end{aligned}$$

$$\begin{aligned}\nexists \underline{U(f, P_\varepsilon)} &= \sum_{j=1}^3 M_j |I_j| \\ &= 1 \times \left(\frac{1}{2} - \varepsilon\right) + 1 \times 2\varepsilon + 0 \\ &= \frac{1}{2} + \varepsilon.\end{aligned}$$

$$\therefore \underline{U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon.}$$

$$\text{i.e. } U(f, P_\varepsilon) - L(f, P_\varepsilon) < 3\varepsilon.$$

why? $\Rightarrow f \in R[0, 1].$

$$\underline{Q:} \quad \int_0^1 f = ?$$

$$\begin{aligned}\therefore \text{For } \varepsilon = \frac{1}{2n}, \text{ we have,} \\ U(f, P_n) - L(f, P_n) &= \frac{1}{n} \\ &\downarrow 0 \\ \Rightarrow \lim [U(f, P_n) - L(f, P_n)] &= 0. \\ \therefore U(f, P_n) &= \frac{1}{2} + \frac{1}{2n} \\ \Rightarrow \int_0^1 f &= \frac{1}{2}.\end{aligned}$$

Remark: In fact, if $f \in B[0, 1]$ with finitely many discontinuity, then $f \in R[0, 1].$

— WAIT —

Let's make it more refined!!

The theory

Def: Let $P: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. ~~For~~ A fr.

$T_P: \{I_j\}_{j=1}^n \longrightarrow [a, b]$ is called a tag ^{of P} if

$T_P(I_j) \in I_j$ $\forall j=1, \dots, n.$

Simply

Given a partition $P: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$, a

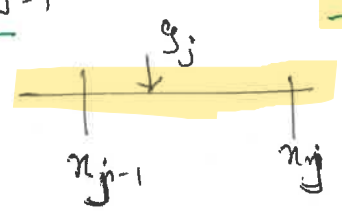
tag or tag set ^{of P} is a collection of points $\{g_j\}_{j=1}^n$

s.t. $g_j \in I_j$ $\forall j=1, \dots, n.$

Even simpler.

of P / for P / corresponding to P.

A tag set $T_P = \{g_j\}_{j=1}^n$ where $g_j \in I_j$ $\forall j=1, \dots, n.$



$P: 0 < \frac{1}{2} < 1 < 2 < 2\frac{1}{2} < 4$

$T_P = \{ \frac{1}{6}, \frac{5}{8}, \frac{4}{3}, 2\frac{1}{5}, 3\frac{1}{3} \}$

a tag set.

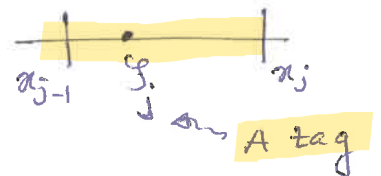
Note: If P has n-nodes [nodes excludes end points $\Rightarrow n+1$ subintervals], then $\#T_P = n+1$ $\forall$ tag set T_P .

Def: Let $f \in \mathcal{B}[a, b]$, $P \in \mathcal{P}[a, b]$. & suppose T_P a tag set of P .

The Riemann sum of f w.r.t. (P, T_P) is defined by:

$$S(f, P) := \sum_{j=1}^n f(y_j) |I_j|, \quad \text{--- } \otimes$$

where: $T_P = \{y_j\}_{j=1}^n$.



Note: ① $S(f, P)$ depends on $T_P = \{y_j\}_{j=1}^n$.

② LHS of $\otimes$ doesn't involve T_P but it is there!!

③ $\exists$ infinitely many tag sets for a given partition P .

④ In fact a question; What is the meaning of Riemann sum?

Fact: (Answering ④)

Suppose $P: a = x_0 < \dots < x_n = b$ be a partition &

$T_P = \{y_j\}_{j=1}^n$ be a tag of P .

$$\therefore m_j \leq f(y_j) \leq M_j \quad \forall j = 1, \dots, n$$

$$\Rightarrow m_j |I_j| \leq f(y_j) |I_j| \leq M_j |I_j|$$

$$\Rightarrow \sum \leq \sum \leq \sum \quad \forall j = 1, \dots, n.$$

$$\Rightarrow L(f, P) \leq S(f, P) \leq U(f, P).$$

T_P $\checkmark$

T_P $\checkmark$

T_P $\checkmark$

$\therefore$ it is a better approximation.

HOWEVER, $S(f, P)$ depends on T_P

but $L(f, P)$ & $U(f, P)$ does not!!

∴ We have:

$$L(f, P) \leq S(f, P) \leq U(f, P) \quad \forall P \in \mathcal{P}[a, b]$$

BUT, this depends on tag sets!!

We predict: $f \in \mathcal{R}[a, b] \iff [\exists \lambda \in \mathbb{R} \text{ s.t. } S(f, P) \rightarrow \lambda \text{ as } \|P\| \rightarrow 0.]$

Meaning? Whatever it is, ~~that~~ we will be in a good situation, ~~as~~ λ doesn't depend on tag set!!

Meaning? $\{ \epsilon > 0, \delta > 0 \}$ s.t. $|f(x) - \lambda| < \epsilon$ $\forall \|x\| < \delta$

$F: \mathcal{P}[a, b] \rightarrow \mathbb{R}$
 $\omega \|P\| \rightarrow 0$
 $F: X \rightarrow \mathbb{R}$
 $F(x) \rightarrow \lambda$
 $\|x\|: X \rightarrow \mathbb{R} \geq 0$

$U_S: \mathcal{P}[a, b] \times \{\text{tag sets}\} \rightarrow \mathbb{R}$
 $G: \mathbb{R} \rightarrow \mathbb{R}$

Def: Given $f \in \mathcal{R}[a, b]$, we say that

$$\lim_{\|P\| \rightarrow 0} S(f, P) = \lambda, \text{ for some } \lambda \in \mathbb{R}, \text{ if}$$

for $\epsilon > 0 \exists \delta > 0$ s.t.

$$|S(f, P) - \lambda| < \epsilon \quad \forall P \in \mathcal{P}[a, b] \text{ s.t. } \|P\| < \delta.$$

Danger: " $\forall T_P$ " is a part of the definition.

Fact: The limit is !.

(! = unique). $\exists \cdot \exists \cdot \rightarrow \leftarrow$

Proof: Suppose $\lim_{\|P\| \rightarrow 0} S(f, P) = \lambda_1$ & $\lim_{\|P\| \rightarrow 0} S(f, P) = \lambda_2$,

for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

If not, let $|\lambda_1 - \lambda_2| := \varepsilon > 0$.

$\therefore \exists \delta_1, \delta_2 > 0$ s.t.

$$|S(f, P) - \lambda_1| < \varepsilon/2 \quad \forall \|P\| < \delta_1$$

$$\& \quad |S(f, P) - \lambda_2| < \varepsilon/2 \quad \forall \|P\| < \delta_2$$

$\therefore$ For $\delta := \min\{\delta_1, \delta_2\}$, we have:

As usual, $\forall$ tags too. $\rightarrow$

$$|S(f, P) - \lambda_1| < \varepsilon/2 \quad \& \quad |S(f, P) - \lambda_2| < \varepsilon/2 \quad \forall \|P\| < \delta.$$

$$\therefore \varepsilon = |\lambda_1 - \lambda_2| \leq |S(f, P) - \lambda_1| + |S(f, P) - \lambda_2|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon \Rightarrow \varepsilon < \varepsilon \rightarrow \text{contradiction}$$

$$\Rightarrow \lambda_1 = \lambda_2 \quad \square$$

And, the good!! :

Thm: Let $f \in \mathcal{B}[a, b]$. Then $f \in \mathcal{R}[a, b] \iff \exists \lambda \in \mathbb{R} \rightarrow$

$$\lim_{\|P\| \rightarrow 0} S(f, P) = \lambda.$$

In this case, $\int_a^b f = \lambda.$

[Note: ① Again, " $S(f, P) \rightarrow \lambda \quad \forall T_P$ ".

② " $\Rightarrow$ " part is more remarkable & effective.

i.e. Suppose we know that $f \in \mathcal{R}[a, b]$.

Alright: Consider $f \in C[a, b]$. ← As for example.

Consider $P_n: a = x_0 < x_0 + h < x_0 + 2h < \dots < x_0 + (n-1)h < x_0 + nh = b$.

Consider the tag set T_P as: $\{x_0 + jh\}_{j=0}^{n-1}$ or $\{x_0 + jh\}_{j=1}^n$

Compute $S(f, P)$.

And, in this case, $\int_a^b f = \lim_{n \rightarrow \infty} S(f, P_n)$ is simply the

limit of "Newton Sums"!!

No sym. in the statement. But we will get in to it soon.

∴ School integ. is justified!!

Proof: "⇒" Let $f \in R[a, b]$. Suppose $\lambda := \int_a^b f$.

Fix $\varepsilon > 0$.

∵ $f \in R[a, b]$, $\exists \delta > 0$ s.t.

$$U(f, P) - L(f, P) < \varepsilon$$

$$\forall P \in \mathcal{P}[a, b]$$

$$\text{s.t. } \|P\| < \delta.$$

Darboux Criterion.

We know that

$$L(f, P) \leq S(f, P) \leq U(f, P)$$

$$\forall P \in \mathcal{P}[a, b]$$

$$\forall \|P\| < \delta,$$

$$\text{Now } U(f, P) < \varepsilon + L(f, P) \leq \varepsilon + \int f = \varepsilon + \lambda.$$

$$L(f, P) > U(f, P) - \varepsilon \geq \int f - \varepsilon = \lambda - \varepsilon.$$

$$\therefore \oplus \Rightarrow \lambda - \varepsilon < S(f, P) < \lambda + \varepsilon \quad \forall \|P\| < \delta.$$

$$\Rightarrow |S(f, P) - \lambda| < \varepsilon$$

$\forall T_P$.
— " —

$$\Rightarrow \lim_{\|P\| \rightarrow 0} S(f, P) = \lambda.$$

" $\Leftarrow$ " Suppose $\lambda := \lim_{\|P\| \rightarrow 0} S(f, P)$ exists.

Let $\varepsilon > 0$.

$\therefore \exists \delta > 0$ s.t.

$$|S(f, P) - \lambda| < \frac{\varepsilon}{3} \quad \forall \|P\| < \delta \quad \text{AND } T_P.$$

$$\Rightarrow \lambda - \varepsilon/3 < S(f, P) < \lambda + \varepsilon/3 \quad \text{--- (1) ---}$$

Recall: $S(f, P) = \sum_{j=1}^n f(\xi_j) (\alpha_j - \alpha_{j-1})$; where $\{\xi_j\} = T_P$
AND $\xi_j \in [\alpha_{j-1}, \alpha_j]$.

- (i) P is fixed. T_P is NOT.
- (ii) A finite sum.
- (iii) f is bounded.



By taking $\inf$ ($\&$ sup) over $\xi_j \in [\alpha_{j-1}, \alpha_j]$
[i.e. $\inf$ $\&$ sup over T_P]

by (1), we have:

$$\left. \begin{aligned} \lambda - \varepsilon/3 &\leq L(f, P) \leq \lambda + \varepsilon/3 \\ \& \quad \lambda - \varepsilon/3 &\leq U(f, P) \leq \lambda + \varepsilon/3 \end{aligned} \right\} \text{--- (2) ---}$$

$$\therefore U(f, P) - L(f, P) \leq \lambda + \varepsilon/3 - (\lambda - \varepsilon/3) = 2\varepsilon/3$$

$$\Rightarrow U(f, P) - L(f, P) < \varepsilon.$$

$\therefore$ Cauchy Criterion $\Rightarrow \underline{f \in R[a, b]}$.

Finally, $(*) \Rightarrow$

$$\lambda - \varepsilon/3 \leq L(f, P) \leq \int_a^b f \leq U(f, P) \leq \lambda + \varepsilon/3.$$

$$\Rightarrow \left| \lambda - \int_a^b f \right| < \varepsilon/3 \quad \forall \varepsilon > 0 \text{ small.}$$

$$\Rightarrow \int_a^b f = \lambda.$$

Useful tool.

Thm: Suppose $f \in R[a, b]$ & $\{P_n\} \subseteq \mathcal{P}[a, b]$ s.t. $\|P_n\| \rightarrow 0$.
 Then $\lim_{n \rightarrow \infty} S(f, P_n) = \int_a^b f$.
 or $\forall$ Tag set T_{P_n} i.e. limit is regardless of the choice of tag sets.

$f \in R[a, b] \Rightarrow \lim_{\|P\| \rightarrow 0} S(f, P) = \int_a^b f$

Not the standard limit

Proof: Let $\varepsilon > 0$. By Darboux criterion, $\exists \delta > 0$ s.t.
 $U(f, P) - L(f, P) < \varepsilon \quad \forall \|P\| < \delta, P \in \mathcal{P}[a, b].$

$\exists \{P_n\} \subseteq \mathcal{P}[a, b]$
 s.t. $\|P_n\| \rightarrow 0$

$$\therefore \|P_n\| \rightarrow 0, \exists N_0 \in \mathbb{N} \text{ s.t.}$$

$$\|P_n\| < \delta \quad \forall n \geq N_0.$$

cc $\therefore U(f, P_n) - L(f, P_n) < \varepsilon \quad \forall n \geq N_0.$

$$\Rightarrow \left[U(f, P_n) - \int_a^b f \right] + \left[\int_a^b f - L(f, P_n) \right] < \varepsilon.$$

$$\therefore \int f = \overline{\int f} = \underline{\int f}$$

$\therefore [-] \geq [-] \geq 0$, it follows that:

$$\left. \begin{aligned} 0 &\leq U(f, P_n) - \int_a^b f < \varepsilon \\ \& \quad 0 \leq \int_a^b f - L(f, P_n) < \varepsilon. \end{aligned} \right\} \forall n \geq N_0.$$

$$\Rightarrow \lim_{n \rightarrow \infty} U(f, P_n) = \int_a^b f$$

$$\& \quad \lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f.$$

Remark:

"..." part is a general fact !!

$f \in R[a, b] \Rightarrow \exists \{P_n\} \subseteq \mathcal{P}[a, b]$

s.t. $\lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n)$

" $\Leftarrow$ " is also true!! $\left(= \int_a^b f \right)$

Finally, since $U(f, P_n) \leq S(f, P_n) \leq L(f, P_n) \forall n$,
by the Squeeze theorem:

$$\lim_{n \rightarrow \infty} S(f, P_n) = \int_a^b f.$$

Again, regardless of tag tags!!



The above result is fair & very useful !!

"School integration" verified & justified.

Consider $f \in C[a, b]$ $\leftarrow$ "A School f_n "

For $n \in \mathbb{N}$, consider $P_n: a = x_0 < x_1 < \dots < x_n = b$ with

$$x_j - x_{j-1} = \frac{b-a}{n} \quad \leftarrow \text{"School partition"}$$

$$\therefore \|P_n\| = \frac{b-a}{n} \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow \|P_n\| \rightarrow 0.$$

Then for any ^{***} tag set $\{ \xi_j \}_{j=1}^n$, we have:

$$\int_a^b f = \lim_{n \rightarrow \infty} \left[\frac{b-a}{n} \sum_{j=1}^n f(\xi_j) \right] \quad \text{--- } \textcircled{*}$$

"The school time $\xi_j := a + \frac{b-a}{n} (j-1)$ $\forall j = 1, \dots, n$

or $\xi_j := a + \frac{b-a}{n} j$ "

end points



The precise "School integration" !!

Remark: Of course $\textcircled{*}$ holds for all $f \in R[a, b]$!!

~~~~~

∴ So far, we have the following (Summary):

Let  $f \in R[a, b]$ . TRUE:

①  $f \in R[a, b]$ .

② [Cauchy criterion]: For  $\varepsilon > 0 \exists P_\varepsilon \in \mathcal{P}[a, b]$  s.t.  
 $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$ .

③ [Darboux criterion]: For  $\varepsilon > 0 \exists \delta > 0$  s.t.

$$U(f, P) - L(f, P) < \varepsilon \quad \forall P \in \mathcal{P}[a, b] \text{ with } \|P\| < \delta.$$

④  $\exists \{P_n\} \in \mathcal{P}[a, b]$  s.t.  $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$ .

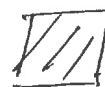
[In this case:  $\lim U(f, P_n) = \lim L(f, P_n) = \int_a^b f$ .]

(5)  $\exists \{P_n\} \subseteq \mathcal{P}[a, b]$  s.t.  $\lim_{n \rightarrow \infty} L(f, P_n) = \lim_{n \rightarrow \infty} U(f, P_n)$ .

both exist.

(6)  $\lim_{\|P\| \rightarrow 0} S(f, P) = \lambda$  exists ( $\forall$  tag set).

In this case:  $\lambda = \int_a^b f$ .



Remark: Recall:  $\mathcal{R}[a, b] := \{f \in \mathcal{B}[a, b] : f \text{ is integrable}\}$ .

Also recall:  $\mathcal{B}[a, b]$  is a vector space.

$\forall f, g \in \mathcal{B}[a, b] \ \forall r \in \mathbb{R},$   
 $f + rg \in \mathcal{B}[a, b]$

Here  $(f + rg)(t) = \underline{f(t)} + r \underline{g(t)}$   
 $\forall t \in [a, b]$

# Also  $\forall f, g \in \mathcal{B}[a, b],$

$fg \in \mathcal{B}[a, b]$ . Why?

Here  $(fg)(x) = f(x)g(x) \ \forall x \in [a, b]$

# Also, if  $[a, b] \xrightarrow{f} [c, d] \xrightarrow{g} \mathbb{R}$   
 are bounded, then  $g \circ f \in \mathcal{B}[a, b] !!$

Why?

Good with  
 compositions.

$f$  is bdd.  $\Rightarrow M > 0$   
 $\forall x \in [a, b] \quad |f(x)| \leq M$   
 $f + rg: [a, b] \rightarrow \mathbb{R}$   
 $\cap \mathcal{B}[a, b] \quad ? (b/w)$   
 "Algebra"



∴ We can ask all the questions for  $\mathcal{R}[a, b]$  [by replacing  $\mathcal{B}[a, b]$ ].

∴  $\mathcal{R}[a, b]$  a vector space? An "algebra"?

SECONDLY:

Consider ~~Suppose~~  $\mathcal{I} : \mathcal{R}[a, b] \rightarrow \mathbb{R}$  defined by

$$\mathcal{I}(f) = \int_a^b f \quad \forall f \in \mathcal{R}[a, b]$$

$M_n(\mathbb{R})$   
 $A \cdot B$   
 $A(B+C) = AB+AC$

We need to think about " $\mathcal{I}$ ",  $\mathcal{R}[a, b]$  & the structure of  $\mathcal{R}[a, b]$  together. Like:

$$\mathcal{I}(f + rg) = \mathcal{I}(f) + r \mathcal{I}(g) \quad ?$$

"Linear".

↑ This is really a natural question.

$$\mathcal{I}(fg) = \mathcal{I}(f) \mathcal{I}(g) \quad ?$$

"multiplicative"

↑ well, no harm in asking!!

If  $f \leq g$  (i.e.  $f(x) \leq g(x) \forall x \in [a, b]$ ),

then  $\mathcal{I}(f) \leq \mathcal{I}(g)$  ?

"Order preserving"

Let  $a < c < b$ , &  $f \in \mathcal{R}[a, b]$ .

$$\stackrel{?}{\Rightarrow} \mathcal{I}(f) = \int_a^c f + \int_c^b f \quad ?$$

splits?

ETC.!!

Thm: Let  $f, g \in R[a, b]$  &  $r \in \mathbb{R}$ . Then

$$f + rg \in R[a, b] \text{ & }$$

$$\int_a^b f + rg = \int_a^b f + r \int_a^b g.$$

[In particular:  $\int f \pm g = \int f \pm \int g$ ;  $\int cf = c \int f$  etc.]

Proof:

Note that  $(f + rg)(x) = f(x) + rg(x) \quad \forall x \in [a, b]$ .

First, we consider  $rf$ .

To prove  $\int rf = r \int f$ .  $\checkmark$

Recall: For  $P \in \mathcal{P}[a, b]$  &  $T_P$  a tag set,

$$S(f, P) = \sum_{j=1}^n f(\xi_j) |I_j|$$

$$P := a = x_1 < \dots < x_n = b.$$

$$\xi_j \in I_j := [x_j, x_{j+1}]$$

$$\forall j = 1, \dots, n$$

$$\& T_P = \{\xi_j\}_{j=1}^n.$$

$$\therefore S(rf, P) = r S(f, P). \quad \leftarrow (\because (rf)(x) = rf(x) \quad \forall x.)$$

$$\Rightarrow \left| S(rf, P) - r \int_a^b f \right| = |r| \left| S(f, P) - \int_a^b f \right| \quad \forall P \in \mathcal{P}[a, b]$$

Presenting one out of many proofs.

$\because f \in R[a, b]$ , the RHS can be made smallest.

$$\Rightarrow rf \in R[a, b] \& \int_a^b rf = r \int_a^b f.$$

$\therefore$  It is now enough to prove that  $\int_a^b f + g = \int_a^b f + \int_a^b g$ .

$$\text{But again: } S(f+g, P) = S(f, P) + S(g, P)$$

$$\begin{aligned} \therefore \left| S(f+g, P) - \int_a^b f - \int_a^b g \right| &= \left| (S(f, P) - \int_a^b f) + (S(g, P) - \int_a^b g) \right| \\ &\leq \left| S(f, P) - \int_a^b f \right| + \left| S(g, P) - \int_a^b g \right| \end{aligned}$$

$\because f, g \in R[a, b]$ ,  $\forall \epsilon > 0 \exists \delta_1, \delta_2 > 0$  s.t.

$$\left| S(f, P) - \int_a^b f \right| < \epsilon/2 \quad \forall \|P\| < \delta_1 \quad \& T_P$$

$$\forall \left| S(g, P) - \int_a^b g \right| < \varepsilon/2 \quad \forall \|P\| < \delta_2 \quad \text{on } T_P.$$

Choose  $0 < \delta < \min\{\delta_1, \delta_2\}$ .

$$\therefore \left| S(f+g, P) - \int_a^b f - \int_a^b g \right| < \varepsilon/2 + \varepsilon/2 \quad \forall \|P\| < \delta \quad \text{on } T_P.$$

$$\Rightarrow f+g \in \mathcal{R}[a, b] \quad \& \quad \int_a^b f+g = \int_a^b f + \int_a^b g.$$

□

# A very Serious H.W.:

$$B = [1, 2] \Rightarrow \hat{B} = [0, 1] \cup [-1, -2]$$

Let  $B \subseteq \mathbb{R}$  be a bdd set. Then  $\hat{B} := \{|x-y| : x, y \in B\}$  is also bdd. Moreover:

$$\sup \hat{B} = \sup B - \inf B. \quad \text{--- HW ---}$$



Fact: Let  $f \in \mathcal{B}[a, b]$ . Then  $\forall P \in \mathcal{P}[a, b]$ , we have

$$M_P - m_P = \sup \left\{ |f(x) - f(y)| : x, y \in [a, b] \right\}$$

$$\& \quad m_j - m_j = \sup \left\{ |f(x) - f(y)| : x, y \in [x_{j-1}, x_j] \right\}.$$

$$\forall j = 1, \dots, n.$$

where:  $P: a = x_0 < \dots < x_n = b$ .

Useful.

often,  $\text{osc } f := \sup_{I_j} \{ |f(x) - f(y)| : x, y \in I_j \}.$

The oscillation of  $f$  on  $I_j$ .

$$j = 1, \dots, n.$$

In general:

Def: For  $A \subseteq \mathbb{R}$  & a bdd  $f: A \rightarrow \mathbb{R}$ , the oscillation of  $f$  is defined by:

$$\text{osc}_A f := \sup_A f - \inf_A f. \quad \checkmark$$

~~$\therefore \text{osc}_{I_j} f = M_j - m_j \quad \forall j$~~  # ~~useful.~~ ~~in general:~~ <sup>true</sup>

Fact: For  $f \in \mathcal{B}[a, b]$  &  $P \in \mathcal{P}[a, b]$ , with  $P: a = x_0 < x_1 < \dots < x_n = b$ ,

We have  $U(f, P) - L(f, P) = \sum_{j=1}^n (M_j - m_j) |I_j|.$

$$\Rightarrow U(f, P) - L(f, P) = \sum_{j=1}^n \text{osc}_{I_j} f |I_j|$$

More arithmetic & properties of  $\mathcal{R}[a, b]$ :

Suppose  $f, g \in \mathcal{R}[a, b]$ . Then:

①  $f + rg \in \mathcal{R}[a, b] \quad \forall r \in \mathbb{R}$  &  $\int_a^b f + rg = \int_a^b f + r \int_a^b g.$

— done. —

②  $f^2 \in \mathcal{R}[a, b]$ .

(But  $\int_a^b f^2 \neq \left(\int_a^b f\right)^2$  : in general

— Why — ? 2 )

Proof: Clearly,  $|f(x)|^2 \leq M^2 \quad \forall x$

$$\Rightarrow f^2 \in \mathcal{B}[a, b].$$

(in fact,  $f^n \in \mathcal{R}[a, b] \quad \forall n \geq 1$ )

Let  $\varepsilon > 0$ .

By Cauchy Criterion,  $\exists P \in \mathcal{P}[a, b]$  s.t.

$$P: a = x_0 < x_1 < \dots < x_n = b.$$

$$0 \leq U(f, P) - L(f, P) < \frac{\varepsilon}{2M} \quad \text{--- } (*)$$

Now  $|f(x)^2 - f(y)^2| = |f(x) + f(y)| |f(x) - f(y)|$   
 $\leq 2M |f(x) - f(y)| \quad \forall x, y \in [a, b].$

$$\Rightarrow \sup_{x, y \in I_j} |f(x)^2 - f(y)^2| \leq 2M \sup_{x, y \in I_j} |f(x) - f(y)|$$

$\Downarrow$   
 $\sup_{I_j} f^2 - \inf_{I_j} f^2$

$\Downarrow$   
 $M_j - m_j = \sup_{I_j} f - \inf_{I_j} f$

$\forall j = 1, \dots, n.$

i.e.:

$$\sup_{I_j} f^2 - \inf_{I_j} f^2 \leq 2M \cdot (M_j - m_j)$$

$\underbrace{\sup_{I_j} f^2 - \inf_{I_j} f^2}_{= \text{osc } f^2 \text{ on } I_j} \leq 2M \cdot \underbrace{(M_j - m_j)}_{= \text{osc } f \text{ on } I_j}$

$\forall j = 1, \dots, n.$

$$\therefore U(f^2, P) - L(f^2, P) = \sum_{j=1}^n \text{osc } f^2 \times |I_j| \leq \sum_{j=1}^n 2M \text{osc } f \times |I_j|$$

For  $f \Delta g$ , if  
 $\text{osc } f \leq \frac{\varepsilon}{2M}$  & if  $g \in R$   
 $\Rightarrow f \in R$

$$\leq 2M \times (U(f, P) - L(f, P))$$

$$\stackrel{\text{by } (*)}{<} 2M \times \frac{\varepsilon}{2M}.$$

$$\Rightarrow U(f^2, P) - L(f^2, P) < \varepsilon.$$

$$\Rightarrow f^2 \in R[a, b]. \quad \square$$

(3)  $fg \in R[a, b].$

[Recall:  $(fg)(x) = f(x)g(x)$ ]

Proof:  $\because fg = \frac{1}{4} [(f+g)^2 - (f-g)^2]$

We have by (1) & (2)  $\Rightarrow fg \in R[a, b].$

④  $|f| \in \mathcal{R}[a, b]$

$\bowtie$

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

$\Sigma \rightarrow \int$

like:  $\left| \sum_{i=1}^n x_i \right| \leq \sum_{i=1}^n |x_i| !!$

Proof:

We know  $||p| - |q|| \leq |p - q| \quad \forall p, q \in \mathbb{R}.$

$$\therefore ||f(x)| - |f(y)|| \leq |f(x) - f(y)| \quad \forall x, y \in [a, b].$$

$\Rightarrow$  Let  $P$  be a partition of  $[a, b]$ .  $\bowtie$   $p: a = x_0 < \dots < x_n = b.$

$$\therefore \otimes \Rightarrow \sup_{I_j} |f| - \inf_{I_j} |f| \leq M_j - m_j = \text{osc}_{I_j} f \quad \checkmark \Rightarrow |f| \in \mathcal{R}[a, b]$$

$$\text{osc}_{I_j} |f| = \sup_{I_j} |f| - \inf_{I_j} |f| \Rightarrow \left( \sup_{I_j} |f| - \inf_{I_j} |f| \right) \times |I_j| \leq (M_j - m_j) \times |I_j|.$$

$$\Rightarrow U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P).$$

$\therefore f \in \mathcal{R}[a, b]$ , it follows that  $|f| \in \mathcal{R}[a, b].$

$\therefore \left| \int f \right| \leq \int |f|$  follows from the next observation.

⑤ If  $f \geq 0$  (i.e.  $f(x) \geq 0 \quad \forall x \in [a, b]$ ), then

$$\int_a^b f \geq 0.$$

$$[f \geq 0 \Rightarrow \int(f) \geq 0.]$$

Proof:  $\therefore f \geq 0, \quad \forall P \in \mathcal{P}[a, b],$

$$L(f, P) \geq 0.$$

$$\Rightarrow \int_a^b f \geq 0. \quad \text{But } \int_a^b f = \int_a^b f \Rightarrow \int_a^b f \geq 0.$$

⑥ If  $f(x) \leq g(x)$  (i.e.  $f \leq g$ )  $\forall x \in [a, b]$ , then  $\int_a^b f \leq \int_a^b g$ .

Proof:  $f \leq g \Rightarrow g - f \geq 0$   $\Rightarrow \int (g - f) \geq 0$    
 $\Rightarrow \int g - \int f \geq 0 \Rightarrow \int g \geq \int f$    
 by ⑤ and ①

⑦  $\left| \int_a^b f \right| \leq \int_a^b |f| \quad \forall f \in \mathcal{R}[a, b].$

Proof: ④  $\Rightarrow |f| \in \mathcal{R}[a, b].$

Now  $-|f| \leq f \leq |f|$ .

Now, ⑥  $\Rightarrow -\int |f| \leq \int f \leq \int |f|$

$\Rightarrow \left| \int f \right| \leq \int |f|.$   $\square$

⑧  $\max\{f, g\}, \min\{f, g\} \in \mathcal{R}[a, b].$

$$\begin{aligned} & \left( \max\{f, g\} \right)(x) \\ &= \max\{f(x), g(x)\} \\ & \quad \forall x \in [a, b] \end{aligned}$$

Proof: Use:  $\max\{p, q\} = \frac{(p+q) + |p-q|}{2}$

$\& \min\{p, q\} = \frac{p+q - |p-q|}{2}$

$\forall p, q \in \mathbb{R}.$

$\&$  all the previous observations.



⑨ Let  $\frac{1}{g} \in \mathcal{B}[a, b]$ . Then  $\frac{f}{g} \in \mathcal{R}[a, b]$ .

Proof: Enough to prove that  $\frac{1}{g} \in \mathcal{R}[a, b]$ .

$$\left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| = \frac{1}{|g(x)| |g(y)|} \times |g(x) - g(y)|$$

$$\leq \tilde{M}^2 \times |g(x) - g(y)|$$

$$\text{where } \tilde{M} := \sup_{x \in [a, b]} \left| \frac{1}{g(x)} \right|.$$

$\therefore$  For  $P: a = x_0 < \dots < x_n = b$  in  $\mathcal{P}[a, b]$ , we have:

$$\text{osc}_{I_j} \frac{1}{g} \leq \tilde{M}^2 \times \text{osc}_{I_j} g \quad \forall j.$$

$$\Rightarrow \frac{1}{g} \in \mathcal{R}[a, b].$$

Feel free to adopt this for simpler proofs of earlier observations.

$F_1 \leq G \leq F_2$   
 $\& F_1, F_2 \in \mathcal{R}[a, b]$   
 $\nexists G \in \mathcal{R}[a, b]$   
 $0 \leq G \leq F$   
 $\& F \in \mathcal{R}[a, b]$

DANGER

Observation (general):

$$\text{If } \text{osc}_{I_j} G \leq \text{osc}_{I_j} F$$

$\forall j \& \forall P \in \mathcal{P}[a, b]$ , then  
 $F \in \mathcal{R}[a, b] \Rightarrow G \in \mathcal{R}[a, b]$ .

$$\Rightarrow G \in \mathcal{R}[a, b] !!$$

eg:  $\sim$  HW  $\sim$

⑩ Let  $a < c < b$ . Then  $f|_{[a, c]} \in \mathcal{R}[a, c]$  &  $f|_{[c, b]} \in \mathcal{R}[c, b]$  &

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

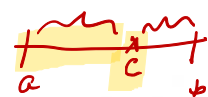
"Try Contradiction: one of the  $f_i$  &  $f_2 \notin \mathcal{R}$ ."



(44)

Proof: Set  $\underline{f_1 = f|_{[a, c]}}$  &  $\underline{f_2 = f|_{[c, b]}}$ . Clearly,  $\underline{f_1 \in \mathcal{R}[a, c]}$  &  $\underline{f_2 \in \mathcal{R}[c, b]}$ .

Let  $\varepsilon > 0$ . By Cauchy Criterion:  $\exists P \in \mathcal{P}[a, b]$  s.t.  
 $U(f, P) - L(f, P) < \varepsilon$ .



WLOG: Let  $c \in P$  (i.e.  $c$  is a node). Otherwise replace the above  $P$  by  $P \cup \{c\}$  & get the same inequality.  
 $\therefore P \cup \{c\} \supset P$ .

Set:  $P: a = x_1 < \dots < x_{m-1} < x_m = c < x_{m+1} < \dots < x_n = b$   
 $\underbrace{\hspace{10em}}_{P_1}$ 
 $\underbrace{\hspace{10em}}_{P_2}$

Then  $P_1: a = x_1 < \dots < x_m = c$  in  $\mathcal{P}[a, c]$   
 &  $P_2: c = x_m < \dots < x_n = b$  in  $\mathcal{P}[c, b]$  }  $\Rightarrow P = P_1 \cup P_2$ .

$$\Rightarrow \underbrace{[U(f_1, P_1) - L(f_1, P_1)]}_{\geq 0} + \underbrace{[U(f_2, P_2) - L(f_2, P_2)]}_{\geq 0} = U(f, P) - L(f, P) < \varepsilon.$$

$$\Rightarrow \# \quad U(f_j, P_j) - L(f_j, P_j) < \varepsilon \quad \forall j = 1, 2, \quad \text{---} \quad (*)$$

$$\Rightarrow \underline{f_1 \in \mathcal{R}[a, c]} \quad \& \quad \underline{f_2 \in \mathcal{R}[c, b]}.$$

$$\text{Set } \lambda_1 := \int_a^c f_1 \quad \& \quad \lambda_2 := \int_c^b f_2 \quad \left( \text{Note: } \lambda_1 = \int_a^c f \quad \& \quad \lambda_2 = \int_c^b f \right).$$

Claim:  $\int_a^b f = \lambda_1 + \lambda_2$ .

$$\int_a^b f \geq L(f, P) = L(f_1, P_1) + L(f_2, P_2) > U(f_1, P_1) + U(f_2, P_2) - 2\varepsilon \geq \lambda_1 + \lambda_2 - 2\varepsilon.$$

by (\*)

$$\int_a^b f \leq U(f, P) = U(f_1, P_1) + U(f_2, P_2) < L(f_1, P_1) + L(f_2, P_2) + 2\varepsilon \leq \lambda_1 + \lambda_2 + 2\varepsilon.$$

by (\*)

$$\therefore \lambda_1 + \lambda_2 - 2\varepsilon \leq \int_a^b f \leq \lambda_1 + \lambda_2 + 2\varepsilon \quad \forall \varepsilon > 0.$$

$$\Rightarrow \underline{\int_a^b f = \lambda_1 + \lambda_2} \quad \text{i.e.} \quad \underline{\int_a^b f = \int_a^c f + \int_c^b f} \quad \square$$

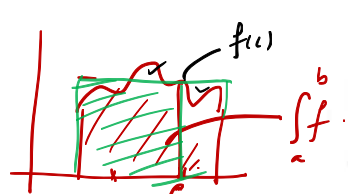
Indeed,  
 $P \in \mathcal{P}[a, b]$   
 as above

&  $\parallel$

Fact: Let  $f \in \mathcal{R}[a, b]$ . Then

$$\textcircled{1} \quad m(b-a) \leq \int_a^b f \leq M(b-a). \quad \leftarrow \text{We know this.}$$

#  $\textcircled{2}$  If in addition,  $f \in \mathcal{C}[a, b]$ , then  $\exists c \in [a, b]$  s.t.

$$f(c) = \frac{1}{b-a} \int_a^b f. \quad \leftrightarrow \quad f(c) \times (b-a) = \int_a^b f. \quad \leftarrow \text{Kind of MVT!!}$$


Proof:  $\because f$  is cont. on  $[a, b]$  ( $\leftarrow$  compact),  $f$  attains all the values in  $[m, M]$ .

$$\text{Then} \quad m \leq \frac{1}{b-a} \int_a^b f \leq M \quad \text{By } \textcircled{1}$$

$$\Rightarrow \exists c \in [a, b] \text{ s.t. } \underline{f(c)} = \frac{1}{b-a} \int_a^b f. \quad \square$$

Def:  $\textcircled{1} \int_a^a f := 0 \quad \forall f: S \rightarrow \mathbb{R}, \quad a \in S$

Fact:  $\textcircled{2} \quad \forall f \in \mathcal{B}[a, b], \quad \int_b^a f := - \int_a^b f. \quad (a < b)$

### Compositions

Thm: Let  $f \in \mathcal{R}[a, b]$ ,  $\text{ran } f \subseteq [c, d]$  & let  $g \in \mathcal{C}[c, d]$ .  
Then  $g \circ f \in \mathcal{R}[a, b]$ .

Proof: Clearly  $g \circ f \in \mathcal{B}[a, b]$ .

$$[a, b] \xrightarrow{f} [c, d] \xrightarrow{g} \mathbb{R}$$

$\xrightarrow{g \circ f}$

$$\left[ \begin{array}{l} \text{Recall: } (g \circ f)(x) \\ = g(f(x)) \\ \forall x \in [a, b]. \end{array} \right]$$

Let  $\varepsilon > 0$ . Since  $g: [c, d] \rightarrow \mathbb{R}$  is unif. cont.

$\exists \delta > 0$  s.t.

$$|g(x) - g(y)| < \frac{\varepsilon}{2(b-a)} \quad \forall |x - y| < \delta.$$

(\*)

Set  $\widetilde{M} := \sup_{y \in [c, d]} |g(y)|$ .

"  $f \in R[a, b]$ , by Cauchy criterion,  $\exists P \in \mathcal{P}[a, b]$

s.t.  $U(f, P) - L(f, P) < \frac{\varepsilon}{4\widetilde{M}}$  (\*\*)

Set  $P: a = x_0 < x_1 < \dots < x_n = b$ .

[Claim:  $U(g \circ f, P) - L(g \circ f, P) < \varepsilon$ .]

~~by (\*)~~

Let  $J = \{1, \dots, n\}$ . I write

$J = J_1 \cup J_2$

disjoint partition.

where:  $J_1 = \{j \in J : M_j - m_j < \delta\}$

$J_2 = \{j \in J : M_j - m_j \geq \delta\}$

$M_j - m_j = \operatorname{osc}_{I_j} f$

$= \sup_{x, y \in I_j} |f(x) - f(y)|$

$\forall j = 1, \dots, n$ ,

Recall:  $M_j - m_j = \operatorname{osc}_{I_j} f$   
 $\forall j = 1, \dots, n$ .

We have: If  $j \in J_1$ , then  $|f(x) - f(y)| < \delta \forall x, y \in I_j$

by (\*):  $|g(f(x)) - g(f(y))| < \frac{\varepsilon}{2(b-a)} \forall x, y \in I_j$

$\Rightarrow \sup_{x, y \in I_j} |g(f(x)) - g(f(y))| < \frac{\varepsilon}{2(b-a)}$

$= \operatorname{osc}_{I_j} g \circ f$

For each  $i = 1, \dots, n$ , we set

$$\tilde{M}_i := \sup_{I_i} g \circ f \quad \& \quad \tilde{m}_i := \inf_{I_i} g \circ f.$$

$$\therefore \forall j \in J_1, \quad \tilde{M}_j - \tilde{m}_j \leq \frac{\varepsilon}{2(b-a)}.$$

$\uparrow$   
 $= \text{osc}_{I_j} g \circ f$

$$\begin{aligned} \Rightarrow \sum_{j \in J_1} (\tilde{M}_j - \tilde{m}_j) \times |I_j| &\leq \frac{\varepsilon}{2(b-a)} \times \sum_{j \in J_1} |I_j| \\ &\leq \frac{\varepsilon}{2(b-a)} \times (b-a) \\ &\leq \varepsilon/2. \end{aligned}$$

All about  $J_1$

Next, we turn to  $J_2$ :

We note that  $\forall j \in J_2$ ,

$$\tilde{M}_j - \tilde{m}_j \leq 2 \tilde{M}_\#$$

$\tilde{M}_\# = \sup_{[a,b]} |g|$

$$\begin{aligned} \therefore \sum_{j \in J_2} (\tilde{M}_j - \tilde{m}_j) \times |I_j| &\leq 2 \tilde{M}_\# \times \sum_{j \in J_2} |I_j| \\ &\leq 2 \tilde{M}_\# \times \sum_{j \in J_2} |I_j| \times \underbrace{\frac{M_j - m_j}{s}}_{\geq 1} \\ &\quad [\because \forall j \in J_2, M_j - m_j \geq s] \end{aligned}$$

$$= 2 \tilde{M}_\# \times \frac{1}{s} \times \sum_{j \in J_2} |I_j| (M_j - m_j)$$

$$\leq 2 \tilde{M}_\# \times \frac{1}{s} \sum_{j \in J} (M_j - m_j) |I_j|$$

$\because J_2 \subseteq J$

$$= \frac{2 \tilde{M}_\#}{s} \times (U(f, P) - L(f, P))$$

$$< \frac{2 \tilde{M}_\#}{s} \times \frac{\varepsilon s}{4 \tilde{M}_\#} \quad (\text{by } \textcircled{xx})$$

$$= \frac{\varepsilon}{2}.$$

$$\therefore U(g \circ f, P) - L(g \circ f, P) = \sum_{j \in J} (\tilde{M}_j - \tilde{m}_j) |I_j|$$

$$= \sum_{j \in J_1} + \sum_{j \in J_2}$$

$\leftarrow \because J = J_1 \cup J_2$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$$\Rightarrow \underline{g \circ f \in \mathcal{R}[a, b]}.$$

$\square$

Cor: Suppose  $f \in \mathcal{R}[a, b]$ . Then:

①  $e^f$ ,  $\sin f(x)$ ,  $\cos f(x)$  are in  $\mathcal{R}[a, b]$ .

②  $\sqrt[n]{f} \in \mathcal{R}[a, b] \quad \forall n \geq 1$  whenever  $f(x) \geq 0$ .

Q: We proved:  $f \in \mathcal{R}, g \in C \Rightarrow g \circ f \in \mathcal{R}$ .

What about: " $f \in \mathcal{R} \ \& \ g \in \mathcal{R} \Rightarrow g \circ f \in \mathcal{R}$ "?

False.

Why??

HW

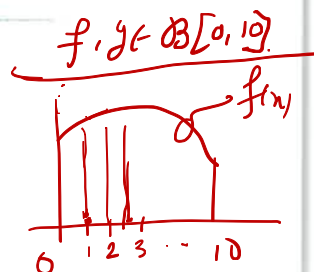
Thm: Suppose  $f, g \in \mathcal{B}[a, b]$  & ~~f and g~~  $f(x) = g(x)$

$\forall x \in [a, b]$  but finitely many. Then  $f \in \mathcal{R}[a, b]$

$$\Leftrightarrow g \in \mathcal{R}[a, b].$$

Moreover, in this case,

$$\int_a^b f = \int_a^b g.$$



$\therefore f = 0$  on  $[a, b]$  except finitely many points

$$\Rightarrow \int_a^b f = 0.$$

$$\int f \neq \int g$$

$$g(x) = \begin{cases} 0 & x \in \mathbb{N} \\ f(x) & x \notin \mathbb{N} \end{cases}$$

Proof: Enough to assume

$$f(x) = g(x) \quad \forall x \in [a, b] \setminus \{c\}$$

$$\& \quad f(c) \neq g(c).$$

for some  $c \in [a, b]$ .

← The general case will then follow by induction.

So, assume the above conditions.

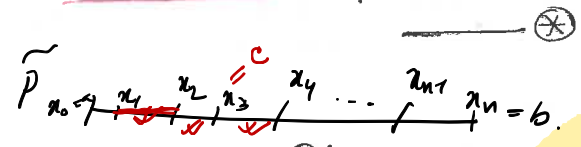
Let  $\varepsilon > 0$  & suppose  $\sup_{[a, b]} |f|, \sup_{[a, b]} |g| \leq \tilde{M}$ .

for some  $\tilde{M} > 0$ .

[ Claim:  $\int f = \int g$  &  $\int f = \int g$  ].

For  $\varepsilon > 0$   $\exists P \in \mathcal{P}[a, b]$  s.t.  $U(f, P) < \int f + \varepsilon/2$ .

Set  $\delta := \frac{\varepsilon}{8\tilde{M}}$ .



Consider a refinement  $\tilde{P}$  of  $P$  s.t.  $\|\tilde{P}\| < \delta$ .

← Always possible.

$\because f = g$  on  $[a, b]$  except  $c \in [a, b]$ ,

$f$  differs from  $g$  on at most 2 subintervals of  $\tilde{P}$ .

[depending on  $c$  being an end point of an subinterval].

Let  $\{\tilde{I}_j\}_{j=1}^n$  be the subintervals of  $\tilde{P}$ .

Assume  $f$  differs from  $g$  on  $I_\ell$ .

Here  $\ell = p$  OR  $\ell = p, p+1$  for some  $p$ .

$\Rightarrow \sup_{I_j} f - \sup_{I_j} g = 0 \quad \forall j \neq p, p+1$

The max 2-point case.  $\& \quad \left| \sup_{I_j} f - \sup_{I_j} g \right| \leq 2\tilde{M} \quad \text{for } j = p, p+1.$

2.

$$\therefore |U(f, \tilde{P}) - U(g, \tilde{P})|$$

$$= \left| \left( \sup_{I_p} f - \sup_{I_p} g \right) \times |I_p| + \left( \sup_{I_{p+1}} f - \sup_{I_{p+1}} g \right) \times |I_{p+1}| \right|$$

$$\leq 2 \times 2 \tilde{M} \times \delta.$$

$$\text{i.e. } |U(f, \tilde{P}) - U(g, \tilde{P})| \leq 4 \delta \tilde{M} = 4 \tilde{M} \times \frac{\varepsilon}{8 \tilde{M}} = \varepsilon/2.$$

$$\text{i.e. } |U(f, \tilde{P}) - U(g, \tilde{P})| \leq \varepsilon/2.$$

So,  $\int g \leq U(g, \tilde{P}) \leq \int f + \varepsilon/2$  (\*\*)

Thus,  $\int g \leq U(g, \tilde{P}) \leq U(f, \tilde{P}) + \varepsilon/2$

↑  
always true.

by (\*\*)

$$\therefore \int g \leq U(f, P) + \varepsilon/2 < \left( \int f + \varepsilon/2 \right) + \varepsilon/2$$

by (\*)

$$\text{i.e. } \int g < \int f + \varepsilon \quad \forall \varepsilon.$$

$$\Rightarrow \int g \leq \int f.$$

Switching  $f$  &  $g$ , this gives:

$$\int f \leq \int g.$$

$$\therefore \int f = \int g.$$

$$\text{Hence } \int f = \int g.$$

$$\therefore \int f = \int f \iff \int g = \int g.$$

□



# §. The fundamental theorem of Calculus (FTC):

Note that if  $\mathcal{D} := \text{diff. fn on } \mathbb{R} / (a, b) / \text{open set};$

then  $d : \mathcal{D} \rightarrow \mathcal{F}(\mathbb{R})$  is a linear map.

$$(df := f').$$

Informal

$$\mathcal{F}(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R}\}.$$

By.  $\mathcal{I} : \mathbb{R}[a, b] \rightarrow \mathcal{F}(\mathbb{R})$  is a linear map.  $x \mapsto \int_a^x f(t) dt$   $f|_{[a, x]} \in \mathbb{R}[a, x]$

FTC ~~even~~ essentially says:

$$\left. \begin{array}{l} \# \quad d \circ \mathcal{I} = \text{identity} \\ \# \quad \mathcal{I} \circ d = \text{identity} \end{array} \right\}$$

TROUBLE: Compositions should be well-defined first!!

In the following, we will explain the informal equalities

~~then~~ then along with all the necessary assumptions, we will make it more formal!!

Def: Let  $S \subseteq \mathbb{R}$  &  $f : S \rightarrow \mathbb{R}$  be a fn. ~~As for this~~

A differentiable fn  $F$  is called an antiderivative or a primitive of  $f$  on  $S$  if

$$f(x) = F'(x) \quad \forall x \in S.$$

eg i)  $\frac{1}{2}x^2$  is an antiderivative of  $x$ .

ii)  $\frac{1}{2}x^2 + c$  ———  $x \quad \forall c \in \mathbb{R}$ .



$$\therefore \underbrace{F}_{\text{antiderivative}} \xrightarrow{\frac{d}{dx}} \underbrace{f}_{\text{function}} \xrightarrow{\frac{d}{dx}} f' \quad \underbrace{\uparrow}_{\text{derivative.}}$$

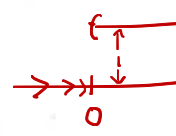
(PROVIDED:  $f$  is diff.)

Q: Do all functions have antiderivatives??

1) polynomials. ✓

2) Continuous fn's. ✓  $\leftarrow$  Why??  $\leftarrow$  WAIT (FTC-II).

3)  $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$

X  $\leftarrow$  Why? 

Ans.

"IVT"

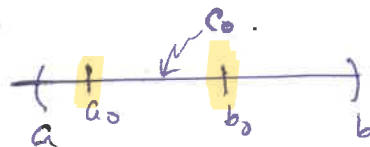
Digression:

Thm: (Darboux's thm)

Let  $f: (a, b) \rightarrow \mathbb{R}$  be a diff. fn. & let  $a < a_0 < b_0 < b$ .  
 If  $f'(a_0) < r < f'(b_0)$ , then  $\exists c_0 \in (a_0, b_0)$  s.t.  
 $f'(c_0) = r$ .

*improves - give result.*

Proof: [Note: If  $f'$  is cont. then this is straight IVT!!]



Set  $g(x) := -f(x) + rx$ .  $x \in (a, b)$ . (Here  $r$  is fixed.)

$$\Rightarrow g \text{ diff. } \& \begin{cases} g'(a_0) < 0 \\ g'(b_0) > 0 \end{cases}$$

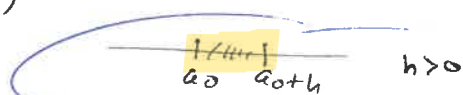
Also  $g|_{[a_0, b_0]}: [a_0, b_0] \rightarrow \mathbb{R}$  Cont. & hence  $g$  on  $[a_0, b_0]$  is unif.

attain its extreme values.

Now  $g'(a_0) > 0 \Rightarrow g(a_0+h) - g(a_0) > 0$  for  $h > 0$  small.

$$\Rightarrow g(a_0) < g(a_0+h)$$

Why??

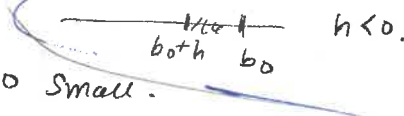


$\Rightarrow g$  does not assume the max. at  $a_0$ .

114  $g'(b_0) < 0 \Rightarrow g(b_0+h) - g(b_0) < 0$  for  $h < 0$  small.

$$\Rightarrow g(b_0) < g(b_0+h) \text{ for } h < 0 \text{ small.}$$

Why??



$\Rightarrow g$  does not assume the max. at  $b_0$ .

$\therefore g$  assumes <sup>(max or local max)</sup> a max at  $c_0 \in (a_0, b_0)$ .

$\because g$  is diff.  $g'(c_0) = 0$ .

$$\Rightarrow f'(c_0) = 0.$$

Note: Thus,  $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$

does not have an antiderivative !!

**ALERT:**

Derivatives need not be continuous !!

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

$f'$  is NOT Cont. at 0.

Thus: Derivative of a  $f_n$  need not be Cont. but still the derivative enjoys the Intermediate Value Property!!

— the novelty of Darboux's thm.

Anyway;

The first FTC

Thm: (FTC-I) Let  $f \in R[a, b]$  &  $F \in C[a, b]$ . Suppose  $F$  is an antiderivative of  $f$  over  $(a, b)$ . Then

$$\int_a^b f = F(b) - F(a).$$

①  $\therefore F'(x) = f(x)$   
 $\forall x \in (a, b)$

② Roughly:  $\int_a^b F' = F(b) - F(a).$

\* [In fact, we can redefine/assign  $F'$  at  $a$  &  $b$ . As  $F'(a, b) = f$  &  $f \in R[a, b]$ , the extended  $F'$  on  $[a, b]$  will be integrable &  $\int_a^b (\text{extended } F') = \int_a^b F'.$ ]

$$\int_{[a, b]} F' = [F] = F(b) - F(a).$$

$\partial[a, b]$   
boundary of  $[a, b] = \{a, b\}$

A remarkable result!!

A continuous analog of sums of differences!!

$$\sum_{j=1}^n (x_j - x_{j-1}) = x_n - x_0.$$

Proof: Let  $P \in \mathcal{P}[a, b]$ .

Set  $P: a = x_0 < x_1 < \dots < x_n = b$ .

$\therefore$  We have the sum of differences:

$$\sum_{j=1}^n (F(x_j) - F(x_{j-1})) = F(b) - F(a).$$

Now  $F|_{[x_{j-1}, x_j]} \in C[x_{j-1}, x_j]$  & diff. on  $(x_{j-1}, x_j)$ .  
 $\forall j=1, \dots, n.$

$\therefore$  MVT  $\Rightarrow \exists \xi_j \in (x_{j-1}, x_j)$  s.t.

$$F(x_j) - F(x_{j-1}) = F'(\xi_j) (x_j - x_{j-1}).$$

$$\Rightarrow F(x_j) - F(x_{j-1}) = f(\xi_j) (x_j - x_{j-1}) \quad \text{--- } (*)$$

as  $F'(x) = f(x) \quad \forall x$   
in  $(a, b)$ .

$$\forall j=1, \dots, n.$$

Now  $\forall j=1, \dots, n$ , we know:

$$m_j (x_j - x_{j-1}) \leq f(\xi_j) (x_j - x_{j-1}) \leq M_j (x_j - x_{j-1}).$$

$$\therefore (*) \Rightarrow m_j (x_j - x_{j-1}) \leq F(x_j) - F(x_{j-1}) \leq M_j (x_j - x_{j-1}) \quad \forall j.$$

$$\Rightarrow \underline{L}(f, P) \leq \sum_{j=1}^n (F(x_j) - F(x_{j-1})) \leq \overline{U}(f, P).$$

$$\text{i.e. } \underline{L}(f, P) \leq F(b) - F(a) \leq \overline{U}(f, P) \quad \forall P \in \mathcal{P}[a, b].$$

$$\Rightarrow \underline{\int} f \leq F(b) - F(a) \leq \overline{\int} f.$$

$$\text{But } f \in \mathcal{R}[a, b]. \Rightarrow \int_a^b f = F(b) - F(a). \quad \square$$

∴ FTC-I  $\Rightarrow \int_a^b f$  can be computed by finding antiderivative of  $f$  !!

Q: How to find (of course, if any!!) an antiderivative?

Ans: "FTC-II".

we know, it may not be!!

The 2<sup>nd</sup> FTC.

# Thm (FTC-II)

Let  $f \in R[a, b]$ . Define  $F(x) := \int_a^x f(t) dt \quad \forall x \in [a, b]$ .

- Then:
- ①  $F \in C[a, b]$ .
  - ② If  $f$  is cont. at  $x_0 \in (a, b)$ , then  $F$  is diff. at  $x_0$  &  $F'(x_0) = f(x_0)$ .
  - ③ If  $f$  is cont. from the right at  $a$ , then  $F'_+(a) = f(a)$ . likewise cont. from left at  $b$ .

Remember? Integration makes  $f$  smoother !!

Proof: ~~Set~~  $M := \sup_{x \in [a, b]} |f(x)|$ . Let  $\epsilon > 0$  &  $x, y \in [a, b]$ .

$$\therefore |F(x) - F(y)| = \left| \int_x^y f(t) dt \right|$$

$$\because -M \leq f(t) \leq M \quad \forall t \in [a, b]$$

$$\left[ \begin{aligned} & \Rightarrow -M(y-x) \leq \int_x^y f(t) dt \leq (y-x)M \quad \forall x, y \in [a, b] \\ & \text{[assuming } y > x \text{]} \end{aligned} \right]$$

∴ the constant fns  $t \mapsto \pm M(y-x)$  are integrable

We have:

(57)

$$-M(y-x) \leq \int_x^y f(t) dt \leq M(y-x). \quad \checkmark$$

$$\Rightarrow \left| \int_x^y f(t) dt \right| \leq M(y-x). \quad [Fov \underline{y > x}],$$

$$\therefore \forall x, y \in [a, b], \quad \left| \int_x^y f(t) dt \right| \leq M|x-y|.$$

$$i.e. \quad |F(x) - F(y)| \leq M|x-y| \quad \forall x, y \in [a, b].$$

$$\Rightarrow F \text{ is uniformly } \text{Cont.} \text{ on } [a, b].$$

— This proves (1).

for (2),

Now, let  $f$  is Cont. at  $x_0 \in (a, b)$ .

$$\therefore \frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(t) dt. \quad \forall x \neq x_0.$$

$$\text{Also, } f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f(x_0) dt.$$

$$\therefore \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x [f(t) - f(x_0)] dt. \quad \forall x \neq x_0$$

Now for  $\varepsilon > 0$   $\exists \delta > 0$  s.t.  $|f(t) - f(x_0)| < \varepsilon$

$\forall |t - x_0| < \delta$ . ← by Cont. of  $f$  at  $x_0$ .

$$\text{Then, } \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \frac{1}{|x - x_0|} \left| \int_{x_0}^x [f(t) - f(x_0)] dt \right|$$

$$\leq \frac{1}{|x - x_0|} \times \int_{x_0}^x |f(t) - f(x_0)| dt.$$

$x_0 - \delta < t < x_0 + \delta$

$$< \frac{1}{|x-x_0|} \times \epsilon \times |x-x_0| \quad \forall |x-x_0| < \delta.$$

$$\Rightarrow \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \epsilon \quad \forall |x - x_0| < \delta.$$

$\Rightarrow$  F is diff. at  $x_0$  &  $F'(x_0) = f(x_0)$ . Q.E.D.

Cor: Let  $f \in C[a, b]$ . Then

$$\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x) \quad \forall x \in [a, b].$$

↑  
differentiable.

D ∘ f = identity on  $C[a, b]$ .

#  $x \sum_{j=1}^n x_j - \sum_{j=1}^{n-1} x_j = x_n$  ↔ Continuous analog of -

↑  
difference of sums!!

Fact: Continuity of  $f$  is <sup>"necessary"</sup> ~~must~~ for diff. of  $x \mapsto \int_a^x f(t) dt$ .

" With  $f \in R[a, b]$ , the cor. may not hold. "

~~with it.~~

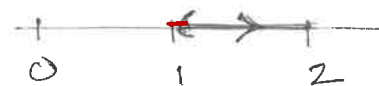
# In particular, if  $f \in C[a, b]$ , then

$$x \mapsto \int_a^x f(t) dt \text{ is an antiderivative of } f.$$

And of course, we know  $\exists f \in R[a, b]$  ~~with~~ with no antiderivatives!!

eg:  $f: [0, 2] \rightarrow \mathbb{R}$  defined by:

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}$$



Clearly,  $f \in \mathcal{R}[0, 2]$ .  $\because f = \chi_{[0, 1]} \in \mathcal{R}[0, 2]$ .

Recall: if  $A \subseteq B$ , then

$\chi_A: B \rightarrow \mathbb{R}$  defined by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

indicator or characteristic fn.

Set  $F(x) := \int_0^x f(t) dt$ .  $\forall x \in [0, 2]$ ,

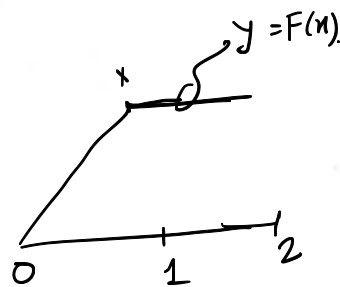
if  $x \in [0, 1]$ , then  $F(x) = \int_0^x 1 \cdot dt = x$

if  $x \in (1, 2]$ , then  $F(x) = \int_0^1 f(t) dt + \int_1^x \underbrace{f(t)}_{=0} dt$ .

$$= \int_0^1 1 \cdot dt + 0.$$

$$= 1 + 0 = 1.$$

$$\therefore F(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 1 & 1 < x \leq 2 \end{cases}$$



$\therefore F$  fails to be ~~cont.~~<sup>diff.</sup> at  $x=1$  (precisely where  $f$  is discont.).

$\therefore$  Integration of an integrable  $f_n$  need not be diff.!!



Thm: (Integration by parts):

Let  $f, g \in \mathcal{D}[a, b]$  &  $f', g' \in \mathcal{R}[a, b]$ . Then

$$\int_a^b f g' + \int_a^b f' g = f(b)g(b) - f(a)g(a).$$

[  $f \in \mathcal{D}[a, b]$  means:  $\exists$  diff. fn  $F$  on  $(a-\varepsilon, b+\varepsilon)$  s.t.  $F|_{[a, b]} = f$ .

OR  $f$  is diff. on  $(a, b)$  &  $f$  has an extension to  $[a, b]$ ;  
(So, extension to only 2 points:  $a$  &  $b$ ).]

*Negligible issue!!*

Proof: Set  $u = fg$ .  $\Rightarrow u' = f'g + fg'$ .

$$\text{FTC} \Rightarrow \int_a^b u' = u(b) - u(a).$$

$$\Rightarrow \int_a^b f g' + \int_a^b f' g = (fg)(b) - (fg)(a) \quad \square$$

$$[\therefore \int_a^b f g' = [fg]_a^b - \int_a^b f' g. \quad \text{the popular form!!}]$$



Thm: (Change of variable): Let  $I \subseteq \mathbb{R}$  be an open interval,

$g: I \rightarrow \mathbb{R}$  diff. &  $g' \in \mathcal{R}(C)$   $\forall$  closed interval  $C \subseteq I$ .

Set  $J = g(I)$ . (Also an interval as  $g$  cont.)

If  $f: J \rightarrow \mathbb{R}$  is cont. &  $a < b$  in  $I$ , then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(x) dx.$$

Proof:  $\because f \circ g \in \mathcal{R}[a, b]$  (as  $f$  cont), we have that  $(f \circ g) g' \in \mathcal{R}[a, b]$

$\therefore f \in \mathcal{C}([g(a), g(b)])$   $F(x) := \int_{g(a)}^x f(t) dt$  is diff. &  $F' = f$  on  $[g(a), g(b)]$

Now  $(f \circ g)'(x) = F'(g(x)) g'(x) = f(g(x)) g'(x) \quad \forall x \in [a, b]$

$$\Rightarrow f \circ g' \in \mathcal{R}[a, b].$$

$$\begin{aligned} \therefore \int_a^b f(g(x)) g'(x) dx &= \int_a^b (f \circ g)'(x) dx = (f \circ g)(b) - (f \circ g)(a) = F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f'(x) dx \quad \square \end{aligned}$$

(61)

Jaydeb Sarkar.

Change of variableThm: Let  $u \in \mathcal{D}[a, b]$ ,  $u' \in \mathcal{R}[a, b]$  &  $f \in \mathcal{C}(u[a, b])$ .

Then

$$\int_a^b f(u(t)) u'(t) dt = \int_{u(a)}^{u(b)} f(x) dx.$$

Note:

$$[a, b] \xrightarrow{u} u[a, b] \xrightarrow{f} \mathbb{R}$$

$$\Rightarrow f \circ u: [a, b] \rightarrow \mathbb{R}$$

The so called  
"u-substitution"

Proof: Note that  $u = \text{constan map} \Leftrightarrow u'(t) \equiv 0$ .

Then the above equality is true (both sides = 0).

$\downarrow$   
" $\because u(a) = u(b)$ "

So, assume that  $u$  is non constant.

$\because f$  is cont,  $f \circ u \in \mathcal{R}[a, b]$ . As  $u' \in \mathcal{R}[a, b]$ , it follows that  $(f \circ u) u' \in \mathcal{R}[a, b]$ .

Also observe that  $u[a, b]$  is an interval. ← Closed?

$$\forall x \in u[a, b], \text{ define } F(x) := \int_{u(a)}^x f(t) dt.$$

By FTC-II,  $F'(x) = f(x) \quad \forall x \in u[a, b]$ .

Then ~~Also~~  $(f \circ u)'(t) = F'(u(t)) u'(t) = f(u(t)) u'(t).$   
 $\forall t \in [a, b]$

$\therefore \text{FTC-I} \Rightarrow$

$$\int_a^b f(u(t)) u'(t) dt = \int_a^b (F \circ u)'(t) dt$$

$$= (F \circ u)(b) - (F \circ u)(a)$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} F'(x) dx.$$

$$= \int_{u(a)}^{u(b)} f(x) dx.$$

$\square$

— x —.

# Improper Integrals.

## Assumptions for Riemann integrations:

$[a, b]$  : i.e. bdd  
& closed interval.

$f \in \mathcal{B}[a, b]$   
 i.e. fn's. must be bdd.

Even a fn is unbdd.,  
 there may be some chance  
 of finite area !!

PRACTICAL  
ISSUES  
 to deal  
 with.

But we really want to  
 integrate over  $(a, \infty)$  or  
 $(-\infty, a)$  or  $(-\infty, \infty)$  etc.

## Improper integrals :

$\int_a^b f$  where  $f$  is unbounded  
 on  $[a, b]$

$\int_a^\infty f$  or  $\int_{-\infty}^b f$  or  $\int_{-\infty}^\infty f$ .

Goal: To find a way to make sense of improper integrals of above types !!

[Notation:  $a, b \in \mathbb{R}$  & always:  $a < b$ ]

## Improper integrals of type I:

Let  $f \notin \mathcal{R}[a, b]$ .

& Suppose  $f \in \mathcal{R}[c, b]$

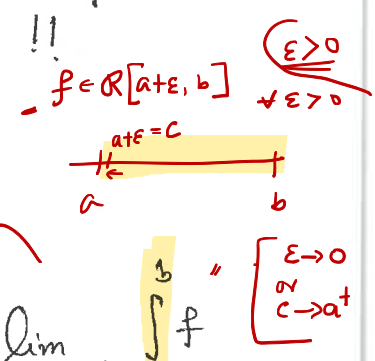
$\forall a < c < b$ . If  $\lim_{c \rightarrow a^+} \int_c^b f$

exists, then we say that the I.I (improper integral)

$\int_a^b f$  Converges & say

$$\int_a^b f = \lim_{c \rightarrow a^+} \int_c^b f \quad \left( \because \int_a^b f = \lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b f \right) \quad \text{--- (A)}$$

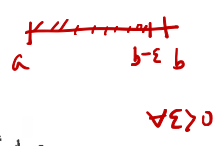
If the limit DNE, then we say  $\int_a^b f$  diverges.



$$\lim_{\epsilon \rightarrow 0^+} \int_{a+\epsilon}^b f$$

|| 4 || let  $f \in R[a, b]$  &  $f \in R[a, c]$   $\forall a < c < b$ .

Then  $\int_a^b f$  converges if



$\lim_{c \rightarrow b^-} \int_a^c f$  exists. We write:

$$\int_a^b f = \lim_{c \rightarrow b^-} \int_a^c f.$$

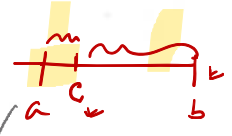
I.I. of type I.

OR, simply:

$$\int_a^b f = \lim_{\epsilon \rightarrow 0^+} \int_a^{b-\epsilon} f. \quad \text{--- (B)}$$

Also

Finally: Let  $a < c < b$ . & suppose  $\int_a^c f$  and/or  $\int_c^b f$  are I.I. of type-I. (~~either~~ (A) or (B)).



We write  $\int_a^b f := \int_a^c f + \int_c^b f$  --- (C)

if both I.I.  $\int_a^c f$  &  $\int_c^b f$  exist.

Otherwise,  $\int_a^b f$  diverges.

|| 4 ||: If  $a < c < b$  &  $f$  is unbounded/discontinuous at  $x = c$ , then we write

$$\int_a^b f := \int_a^c f + \int_c^b f \quad \text{provided both}$$

I.I. in the RHS exist. (A) kind. (B) kind.

eg:

$$(1) \int_0^1 \frac{1}{x^2} dx.$$

Clearly, this is an I.I. of Type - I.

$$\text{Now, } \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{1}{x^2} dx.$$

$$= \lim_{\varepsilon \rightarrow 0^+} \left[ -\frac{1}{x} \right]_{\varepsilon}^1$$

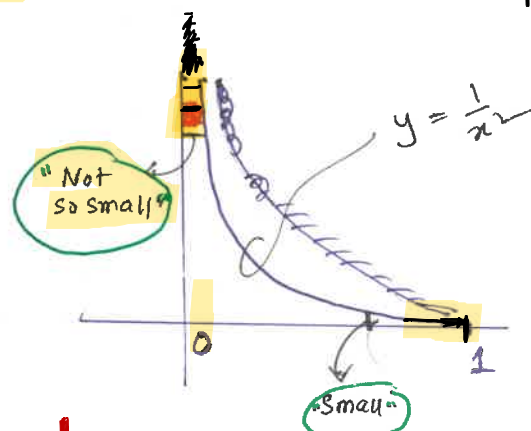
← Why? FTC?

$$= \lim_{\varepsilon \rightarrow 0^+} \left( \frac{1}{\varepsilon} - 1 \right) = +\infty.$$

← why?

$$\therefore \int_0^1 \frac{1}{x^2} dx \text{ diverges.}$$

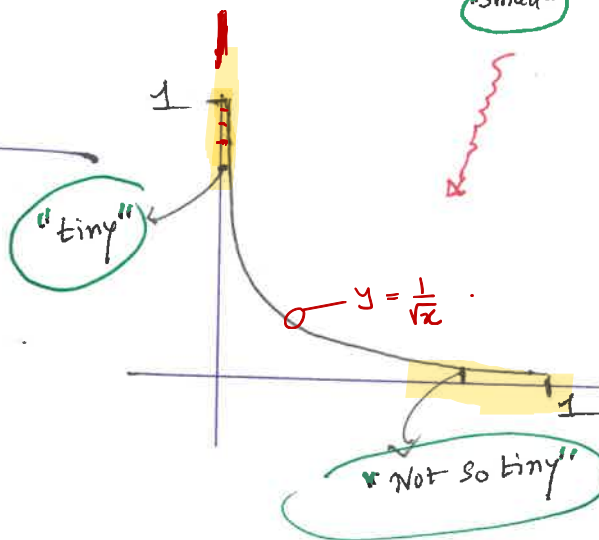
$$\int_a^b f = \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{k=1}^n f\left(a + \frac{k}{n}\right) \right) = \int_a^b f(x) dx$$



(2)

$$\int_0^1 \frac{1}{\sqrt{x}} dx.$$

tells you  
we have  
a chance!!



$$\text{Now } \int_{\varepsilon}^1 \frac{1}{\sqrt{x}} dx$$

$$= 2 \times \left[ x^{1/2} \right]_{\varepsilon}^1 = 2(1 - \sqrt{\varepsilon}).$$

$$\therefore \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{1}{\sqrt{x}} dx = \lim_{\varepsilon \rightarrow 0^+} 2(1 - \sqrt{\varepsilon}) = 2.$$

$\therefore \int_0^1 \frac{1}{\sqrt{x}} dx$  is Convergent ✓

$$\int_0^1 \frac{1}{\sqrt{x}} dx = 2.$$

③  $\int_0^2 \frac{1}{2x-x^2} dx,$

$x \mapsto \frac{1}{2x-x^2}$  is not defined at  $x=0, 2$ .

$\therefore$  We need to investigate

$$\int_0^1 f + \int_1^2 f.$$



OR any  
 $0 < c < 2$ .

Now  $\int_1^2 f = \lim_{\varepsilon \rightarrow 0^+} \int_1^{2-\varepsilon} \frac{1}{x(2-x)} dx.$

$$\left| \frac{1}{x(2-x)} \right| = \left( \frac{1}{x} + \frac{1}{2-x} \right) \frac{1}{2}$$

$$= \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \int_1^{2-\varepsilon} \left( \frac{1}{x} + \frac{1}{2-x} \right) dx.$$

$$= \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \left[ \ln \left( \frac{x}{2-x} \right) \right]_1^{2-\varepsilon}$$

$$= \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \ln \left( \frac{2-\varepsilon}{\varepsilon} \right) = \infty \quad \leftarrow \text{why?}$$

$$\Rightarrow \int_0^2 \frac{1}{2x-x^2} dx \text{ diverges .}$$

④ P.T.  $\int_0^1 \frac{dx}{\sqrt{1-x}} = 2.$  (HW)

In general:

Suppose  $p > 0$ .

Then  $\int_0^1 \frac{dx}{x^p} = ??$

← Type I

Note that  $\forall 0 < \varepsilon < 1$

$$\int_{\varepsilon}^1 \frac{1}{x^p} dx = \begin{cases} \left[ \frac{x^{-p+1}}{-p+1} \right]_{\varepsilon}^1 & \text{if } p \neq 1 \\ [\log x]_{\varepsilon}^1 & \text{if } p = 1. \end{cases}$$

$$= \begin{cases} \frac{1}{1-p} x (1 - \varepsilon^{1-p}) & \text{if } p \neq 1 \\ -\log \varepsilon & \text{if } p = 1 \end{cases}$$

$$\therefore \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{1}{x^p} dx = \begin{cases} \frac{1}{1-p} & 0 < p < 1 \\ \infty & \text{if } p \geq 1. \end{cases}$$

$\therefore$  If  $p > 0$ , then the I.I.  $\int_0^1 \frac{dx}{x^p}$  Converges

for  $0 < p < 1$  & diverges for  $p \geq 1$ .

[33:  $p < 0$  Case is not worth it!!

- why? ]





Natural  
classification.

~~Recall:~~

Thm: Let  $\int_a^b f$  be an I.I. at  $b$ . Then  $\int_a^b f$  Converges  $\iff$  for

$\varepsilon > 0 \exists \delta > 0$  s.t.

$$\left| \int_c^d f \right| < \varepsilon \quad \forall c, d \text{ s.t. } b - \delta < c < d < b.$$

Proof:

" $\Rightarrow$ " Let  $\int_a^b f$  Converges.

Set  $F(x) := \int_a^x f(t) dt. \quad x \in [a, b).$

By assumption:  $\lim_{x \rightarrow b^-} F(x)$  exists.

Recall:  
 $f \in R[a, b - \varepsilon]$   
 $\forall \varepsilon > 0$ . small.

$\therefore$  For  $\varepsilon > 0, \exists \delta > 0$  s.t.

$$|F(c) - F(d)| < \varepsilon \quad \forall c, d \text{ s.t. } b - \delta < c < d < b.$$

$$\underbrace{\int_c^d f}_{\text{" "}}$$

$$\Rightarrow \left| \int_c^d f \right| < \varepsilon$$

" $\Leftarrow$ "  $\otimes$  holds  $\Rightarrow \lim_{x \rightarrow b^-} F(x)$  exists.

$$\Rightarrow \lim_{x \rightarrow b^-} \int_a^x f \text{ exists.}$$

$\square$

Thm: (Comparison test - I)

Let  $0 \leq f(x) \leq g(x) \quad \forall x \in [a, b)$ . Assume that

$\int_a^b f$  &  $\int_a^b g$  are I.I. at  $b$ .

① If  $\int_a^b g$  converges, then  $\int_a^b f$  converges.

② If  $\int_a^b f$  diverges, then  $\int_a^b g$  diverges.

Proof:  $\because$  ①  $\Rightarrow$  ②, we only prove ①.

Set  $F(x) := \int_a^x g$ .  $\forall x \in [a, b)$ .

$\because g \geq 0$ ,  $F \uparrow$  on  $[a, b)$ .

$\because \int_a^b g$  converges, we have:

$$\int_a^b g = \lim_{x \rightarrow b^-} F(x)$$

$$= \sup \left\{ F(x) : x \in [a, b) \right\}$$

$$\quad \quad \quad \underbrace{\quad}_\text{" "}$$

$$\quad \quad \quad \int_a^x g(x).$$

$$= \sup \left\{ \int_a^x g(x) : x \in [a, b) \right\}.$$

Now  $0 \leq f(x) \leq g(x) \quad \forall x \in [a, b)$

$$\Rightarrow 0 \leq \int_a^x f \leq \int_a^x g. \quad \forall x \in [a, b).$$

$$\leq \int_a^b g.$$

$$\therefore x \mapsto \int_a^x f \uparrow \text{ on } [a, b)$$

$$\& \quad 0 \leq \int_a^x f \leq \int_a^x g \leq \int_a^b g.$$

$$\text{Then} \quad 0 \leq \int_a^x f \leq \int_a^b g \quad \& \quad x \mapsto \int_a^x f \uparrow$$

$$\Rightarrow \lim_{x \rightarrow b^-} \int_a^x f \text{ exists} \quad \& \quad \int_a^b f \leq \int_a^b g.$$

□

(eg:)

$$\text{For } p > 0, \text{ Consider } \int_0^{\pi/2} \frac{\sin x}{x^p} dx.$$

$$\text{We know } x \mapsto \frac{\sin x}{x} \text{ is bdd} \quad \&$$

$$\frac{\sin x}{x} \leq 1 \quad \forall 0 < x \leq \pi/2$$

$$\Rightarrow \frac{\sin x}{x^p} \leq \frac{1}{x^{p-1}} \quad \text{---||---}$$

$$\text{Now } \int_0^{\pi/2} \frac{1}{x^{p-1}} dx \text{ converges only when } p-1 < 1$$

$$\Leftrightarrow p < 2.$$

$\therefore$  By Comparison test:

$$\int_0^{\pi/2} \frac{\sin x}{x^p} dx \text{ converges for } p < 2$$

$$\& \text{ diverges for } p \geq 2.$$

□

Thm: (Limit Comparison test - I)



Let  $f(x), g(x) \geq 0 \quad \forall x \in [a, b]$ .

Suppose  $\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = l$ .

If  $l \neq 0, \infty$ , then  $\int_a^b f$  &  $\int_a^b g$  Converge or diverge together at b.

Proof:

Suppose  $l > 0$ . Choose  $\varepsilon > 0$  small s.t.  $l - \varepsilon > 0$ .

$\therefore \lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = l, \quad \exists c \in [a, b]$  s.t.

$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon \quad \forall x \in (c, b).$$

$$\Rightarrow l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \quad \forall x \in (c, b).$$

$$\Rightarrow (l - \varepsilon) g(x) < f(x) < (l + \varepsilon) g(x).$$

$\therefore l - \varepsilon > 0$  &  $g(x) \geq 0 \quad \forall x$ , it follows that

$$0 \leq (l - \varepsilon) g(x) \leq f(x) \quad \forall x \in (c, b).$$

$\therefore$  By Comparison test, if  $\int_a^b f$ , or equivalently, if

$\int_c^b f$  Converges at b, then  $(l - \varepsilon) \int_c^b g$ , or equivalently,

$$\underline{(l - \varepsilon) \int_a^b g \text{ Converges at b.} \Rightarrow \int_a^b g \text{ Converges at b.}}$$

Now suppose  $\int_a^b g$ , or equiv.,  $\int_c^b g$  converges at  $b$ .

Again by  $f(x) < (1+\epsilon)g(x) \quad \forall x \in (c,b)$  &  
by the comparison test, it follows that  $\int_a^b f$  converges at  $b$ .

$\therefore \int_a^b f$  converges at  $b \iff \int_a^b g$  converges at  $b$ .

||||;  $\int_a^b f$  diverges at  $b \iff \int_a^b g$  diverges at  $b$ .



eg: (1)  $\int_0^1 \frac{\sin x}{x^2} dx$ .  $\leftarrow$  An I.I. at  $x=0$ .

Set  $f(x) = \frac{\sin x}{x^2}$  &  $g(x) = \frac{1}{x}$ .

$$\therefore \frac{f(x)}{g(x)} = \frac{\sin x}{x}$$

Now  $\lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 > 0$ .

But  $\int_0^1 g = \int_0^1 \frac{1}{x} dx$  diverges  $\leftarrow$  why?

$\therefore$  By Limit comparison test,  $\int_0^1 \frac{\sin x}{x^2} dx$  diverges.

(2)  $\int_0^1 \frac{e^{\sqrt{x}} - 1}{x} dx$ .  $\leftarrow$  I.I. at  $x=0$ .

Set  $f(x) = \frac{e^{\sqrt{x}} - 1}{x}$  &  $g(x) = 1/\sqrt{x}$ .

$$\therefore \frac{f(x)}{g(x)} = \frac{\sqrt{x}(e^{\sqrt{x}} - 1)}{x} = \frac{\sqrt{x}(\sqrt{x} + \text{h.o.t.})}{x}$$

$\rightarrow 1$  as  $x \rightarrow 0^+$ .

why?

$\int \frac{1}{x^p}$  ,  ~~$\frac{1}{x}$~~   
 1

$\therefore \int_0^1 \frac{1}{\sqrt{x}} dx$  Converges, by Limit-Comparison test, (73) Jaydeb Sarkar.  
 $\int_0^1 \frac{e^{\sqrt{x}} - 1}{x} dx$  Converges. 14

(3)  $\int_1^2 \frac{x^2 + x + 1}{(x^2 - 1)^{1/3}} dx.$

$f(x) := \frac{x^2 + x + 1}{(x^2 - 1)^{1/3}}$

~~Rational fn.~~

$g(x) := \frac{1}{(1-x)^{1/3}}$

$\therefore$  We can deal with  $\frac{1}{x+1}$  factor.

$\therefore \lim_{x \rightarrow 1^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 1^+} \frac{(x^2 + x + 1)}{(x+1)^{1/3}} = \frac{3}{2^{1/3}} > 0.$

Now  $\int_1^2 g = \int_1^2 \frac{1}{(1-x)^{1/3}} dx$

$= \int_0^1 \frac{1}{x^{1/3}} dx$   $\leftarrow \begin{cases} 1-x \mapsto x \\ \Rightarrow 1 \rightarrow 0 \\ \quad 2 \rightarrow 1 \end{cases}$

$\therefore p = \frac{1}{3} < 1, \int_1^2 g$  Converges.

$\Rightarrow \int_1^2 \frac{x^2 + x + 1}{(x^2 - 1)^{1/3}} \underline{\text{Converges.}}$  14

Def: An I.I.  $\int_a^b f$  is said to be absolutely convergent if the I.I.  $\int_a^b |f|$  is convergent.

Thm: Absolute Convergent  $\Rightarrow$  Convergent.

Q: " $\Leftarrow$ "? ??

Proof: Let  $\int_a^b f$  be an I.I. at  $x=a$ .

$$\text{Now } -|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in (a, b].$$

$$\int_{-1}^1 \frac{1}{x} dx$$

$$\Rightarrow 0 \leq f(x) + |f(x)| \leq 2|f(x)| \quad \forall x \in (a, b].$$

$\therefore$  the I.I.  $\int_a^b |f|$  is absolutely conv. at  $a$ , by comparison test, it follows that

$$\int_a^b (f(x) + |f(x)|) dx \text{ converges.}$$

Finally,  $\forall a < c < b$ , we have:

$$\begin{aligned} \int_c^b f &= \int_c^b ((f(x) + |f(x)|) - |f(x)|) dx \\ &= \int_c^b (f(x) + |f(x)|) dx - \int_c^b |f(x)| dx \end{aligned}$$

(if both I.I. exist)

$\therefore \int_a^b (f(x) + |f(x)|) dx$  &  $\int_a^b |f(x)| dx$  both exist, it follows that

$\int_a^b f$  also exists.



eg: (Recall from page 72).

$$\int_0^1 \frac{\sin x}{x^2} \cdot dx.$$

$$\therefore \left| \frac{\sin x}{x^2} \right| \not\leq |\sin x| \quad \forall x \in (0, 1]$$

$\int_0^1 |\sin x| dx$  is convergent, it follows that  $\int_0^1 \frac{\sin x}{x^2}$  is A.C. & hence convergent.



Def: (I.I. of type II):

Fix  $a \in \mathbb{R}$  & suppose  $f \in \mathcal{R}[a, b] \quad \forall b > a$ .

If  $\lim_{r \rightarrow +\infty} \int_a^r f$  exists, then we say that

$\int_a^{+\infty} f$  converges & we write:

$$\int_a^{+\infty} f = \lim_{r \rightarrow +\infty} \int_a^r f.$$

If  $\lim_{r \rightarrow +\infty} \int_a^r f$  diverges, then we say  $\int_a^{+\infty} f$  diverges.



114  $\int_{-\infty}^b f = \lim_{r \rightarrow -\infty} \int_r^b f$  if Converges.

Assumes  $f \in R[-r, b]$   $\forall r < b$   $r > 0$

Def:  $\int_{-\infty}^{\infty} f := \int_{-\infty}^c f + \int_c^{\infty} f$

— whenever the R.

Def: Let  $f \in R[a, b]$   $\forall a < b$  in  $\mathbb{R}$ . If  $\exists c \in \mathbb{R}$  s.t.

both  $\int_{-\infty}^c f$  &  $\int_c^{\infty} f$  Converge, then we say that  $\int_{-\infty}^{\infty} f$

Converges & write

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^c f + \int_c^{\infty} f \quad (*)$$

HW: If  $\int_{-\infty}^{\infty} f$  Converges, then  $(*)$  is independent of the  
Choice of  $c \in \mathbb{R}$ .

eg: ①  $\int_0^{\infty} \sin x \, dx$

Let  $f(x) = \sin x$ .  $x \in [0, \infty)$ .

$1 - 1 + 1 - 1 + \dots = 0?$  x

Note that  $f \in R[0, R]$   $\forall R \geq 0$ .

$$\therefore \lim_{R \rightarrow \infty} \int_0^R f = \lim_{R \rightarrow \infty} \int_0^R \sin x \, dx$$

$$= \lim_{R \rightarrow \infty} [1 - \cos R]$$

But this limit DNE!  $\Rightarrow \int_0^{\infty} \sin x \, dx$  diverges.

OR Does not Converge.

②  $\int_{-\infty}^0 e^{-x} dx.$

Again,  $e^{-x} \in \mathcal{R}[-R, 0] \quad \forall R > 0.$

Now, for  $R > 0,$

$$\int_{-R}^0 e^{-x} dx = \int_0^R e^x dx \quad [x \rightarrow -x]$$

$$= e^R - 1.$$

$$\therefore \lim_{R \rightarrow \infty} \int_{-R}^0 e^{-x} dx = \lim_{R \rightarrow \infty} (e^R - 1) = \infty.$$

$$\therefore \int_{-\infty}^0 e^{-x} dx \text{ diverges.}$$

③  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$

We observe:  $\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{R \rightarrow \infty} \int_0^R \frac{dx}{1+x^2}$

$$= \lim_{R \rightarrow \infty} [\tan^{-1}(R) - \tan^{-1}(0)].$$

Why?  $\Rightarrow \pi/2 - 0 = \pi/2.$

Also,  $\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{R \rightarrow \infty} \int_{-R}^0 \frac{dx}{1+x^2}$

$$= \lim_{R \rightarrow \infty} \int_0^R \frac{dx}{1+x^2} = \pi/2.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

This is an even fcn.

[14]

Fact: (HW). Let  $a > 0$ . Then

$$\int_a^{\infty} \frac{1}{x^p} dx$$

Converges  $\forall p > 1$  & diverges  $\forall p \leq 1$ .

# Notation: For a fixed  $a \in \mathbb{R}$ , we say  $f \in \mathcal{R}[a, \infty)$  if  $f \in \mathcal{R}[a, R]$   $\forall R > a$ .

Thm. (Comparison test - II).

Let  $a \in \mathbb{R}$ ,  $f, g \in \mathcal{R}[a, \infty)$  & let

$$0 \leq f(x) \leq g(x) \quad \forall x \in [a, \infty).$$

**CAUTION !!**

We are not saying that  $f$  is Riemann integrable.

(i) If  $\int_a^{\infty} g$  Converges, then  $\int_a^{\infty} f$  Converges.

(ii) If  $\int_a^{\infty} g$  diverges, then  $\int_a^{\infty} f$  diverges.

Proof:  $\because 0 \leq f(x) \leq g(x) \quad \forall x \in [a, \infty)$  &

$f, g \in \mathcal{R}[a, \infty)$ , it follows that

$$0 \leq \int_a^t f(x) dx \leq \int_a^t g(x) dx \quad \forall t > a.$$

$$\text{Set } F(t) := \int_a^t f(x) dx \quad \& \quad G(t) := \int_a^t g(x) dx. \quad \forall t > a.$$

$\therefore F, G \in \mathcal{C}(a, \infty)$  & monotonically increasing.

The result now follows immediately.

(Cont. of  $F$  &  $G$  is not needed).

# Examples of Type-II Comparison test :

eg: ①  $\int_a^\infty \frac{dx}{e^x + 1}$

Note that  $0 < \underbrace{\frac{1}{e^x + 1}}_{f(x)} \leq \underbrace{\frac{1}{e^x}}_{g(x)} \quad \forall x \in \mathbb{R}$ .

Now  $\int_0^\infty \frac{1}{e^x} dx$ , & hence  $\int_a^\infty \frac{1}{e^x} dx$  Converges  $\forall a \in \mathbb{R}$ .  
Error page

$$\left[ \because \lim_{R \rightarrow \infty} \int_0^R \frac{1}{e^x} dx = \lim_{R \rightarrow \infty} \int_0^R e^{-x} dx = \lim_{R \rightarrow \infty} (1 - e^{-R}) = 1 \right]$$
$$\Rightarrow \int_0^\infty \frac{1}{e^x} dx = 1$$

$$\Rightarrow \int_a^\infty \frac{dx}{e^x + 1} \text{ Converges (by the Comparison test).}$$

\* ②  $\int_0^\infty e^{-x^2} dx$

Asking for Convergency.  
[Euler-Poisson integral & value =  $\sqrt{\pi/2}$ ]

Note that  $f(x) := e^{-x^2}$  is in  $\underbrace{C[0, R]}_{\hookrightarrow C[0, \infty)}$ ,  $\forall R > 0$ .

Now  $e^{x^2} > x^2 \quad \forall x \in \mathbb{R}$ . Why?

$$\Rightarrow 0 < \frac{1}{e^{x^2}} < \frac{1}{x^2} \quad \forall x \neq 0.$$

Now  $\int_1^\infty \frac{1}{x^2} dx$  Converges. (p=2 > 1 Case)

$\therefore$  by Comparison test,  $\int_1^\infty \frac{1}{e^{x^2}} dx$  Converges.

$$\Rightarrow \int_0^\infty \frac{1}{e^{x^2}} dx \text{ Converges } [\because e^{-x^2} \in C[0, \infty)]$$

Thm: (Limit Comparison test - II):

Suppose  $f, g \in R[a, \infty)$  &  $f(x), g(x) \geq 0 \quad \forall x \in [a, \infty)$ .

If  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = l > 0$ , then  $\int_a^\infty f$  &  $\int_a^\infty g$

Converge or diverge together.

← (proof is similar to Type-I case)

Proof: Fix  $\varepsilon > 0$  s.t.  $l - \varepsilon > 0$ . [ $\because l > 0$ ]

$\because \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = l, \quad \exists M > 0$  s.t.

$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon \quad \forall x > M.$$

$$\Rightarrow (l - \varepsilon) \leq \frac{f(x)}{g(x)} < l + \varepsilon \quad \forall x > M$$

$$\Rightarrow (l - \varepsilon) g(x) < f(x) < (l + \varepsilon) g(x) \quad \forall x > M.$$

Suppose  $\int_a^\infty f$  Converges.

Since  $(l - \varepsilon) g(x) \geq 0 \quad \forall x > M$

$\Rightarrow \int_a^\infty (l - \varepsilon) g$  Conv.  
Comparison Test

$\Rightarrow \int_a^\infty g$  Converges.

If  $\int_a^\infty f$  diverges, then

$$f(x) < (l + \varepsilon) g(x)$$

$$\Rightarrow \frac{1}{l + \varepsilon} f(x) < g(x) \quad \forall x > M$$

$\Rightarrow \int_a^\infty g$  diverges.  
Comparison Test

~~$\therefore \int_a^\infty f$  Conv.  $\Leftrightarrow \int_a^\infty g$  Conv.~~ The rest is HW.  
by divergence part.

eg:  $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}}$

Let  $f(x) = \frac{1}{x\sqrt{x^2+1}}$  &  $g(x) = \frac{1}{x^2}$ .

$\therefore f(x), g(x) \geq 0 \quad \forall x \in [1, \infty)$ .

Now  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+1}}$   
 $= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$   
 $= 1 > 0.$

Note that  $f(x) = \frac{1/x^2}{\sqrt{1+\frac{1}{x^2}}}$

Note:  $f, g \in R[1, \infty)$

Now  $\int_1^{\infty} g = \int_1^{\infty} \frac{1}{x^2} dx$  Converges as  $p=2 > 1$ .

$\therefore$  By Comparison test,  $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+1}}$  Converges.  $\square$

Thm: Let  $f \in R[a, \infty)$ . If  $\int_a^{\infty} |f|$  Converges (i.e.  $\int_a^{\infty} f$  is absolutely convergent), then  $\int_a^{\infty} f$  Converges.

AC  $\Rightarrow$  Converges

Proof: (Very similar to  $\int_a^b f$  case i.e. type I case):

~~$\int_a^b f(x) dx$~~

We have  ~~$\int_a^b f(x) dx$~~

$-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in [a, \infty)$

$\Rightarrow 0 \leq f(x) + |f(x)| \leq 2|f(x)|$

$\therefore \int_a^{\infty} |f|$  Converges, by Comparison test,  $\int_a^{\infty} (f + |f|)$  Converges. Hence  $\int_a^{\infty} f (= \int_a^{\infty} (f + |f|) - \int_a^{\infty} |f|)$  Converges.  $\square$

Q: Converges  $\Rightarrow$  AC?

Some useful integral tests:

We assume  $f \in R[a, \infty)$ .

Thm. (Cauchy's test):  $\int_a^\infty f$  Converges  $\Leftrightarrow \forall \varepsilon > 0 \exists M_0 > 0$

s.t.  $\left| \int_{R_1}^{R_2} f \right| < \varepsilon \quad \forall R_1, R_2 > M_0.$

Recall (Cauchy's limit criterion):  $\lim_{x \rightarrow a} f(x)$  exists  $\Leftrightarrow \forall \varepsilon > 0$

$\exists \delta > 0$  s.t.  $|f(x_1) - f(x_2)| < \varepsilon \quad \forall x_1, x_2 \in (a - \delta, a + \delta) \setminus \{a\}.$

Proof: ~~or~~  $\forall x \geq a$ , define  $F(x) := \int_a^x f(t) dt.$

② ~~def.~~ We say  $\lim_{x \rightarrow \infty} f(x) = l \in \mathbb{R}$  if  $\forall \varepsilon > 0 \exists M_0 > 0$

s.t.  $|f(x) - l| < \varepsilon \quad \forall x > M_0.$

③ (Cauchy's criterion):  $\lim_{x \rightarrow \infty} f(x) = l$  exists  $\Leftrightarrow \forall \varepsilon > 0$

$\exists M_0 > 0 \exists \cdot |f(x_1) - f(x_2)| < \varepsilon \quad \forall x_1, x_2 > M_0.$

— HW —

Proof: Note that  $\int_a^\infty f = \lim_{R \rightarrow \infty} \int_a^R f(t) dt. \quad \text{--- } \otimes$

By defn.  
(if exists)

~~So~~  $F(x) = \int_a^x f(t) dt. \quad \forall x \geq a.$

$\therefore \otimes$  exists  $\Leftrightarrow \forall \varepsilon > 0 \exists M_0 > 0$  s.t.

$\left| \int_a^{R_1} f(t) dt - \int_a^{R_2} f(t) dt \right| < \varepsilon \quad \forall R_1, R_2 > M_0.$

The Cauchy Criterion  $\rightarrow$

$= \left| \int_{R_1}^{R_2} f \right|$

$\Leftrightarrow \left| \int_{R_1}^{R_2} f \right| < \varepsilon \quad \forall R_1, R_2 > M_0.$

□

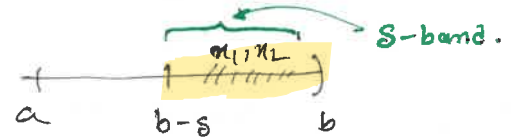
## Deviation: on Page 68

Cauchy's Criterion: Suppose  $f: (a, b) \rightarrow \mathbb{R}$  be a fn. Then

$\lim_{x \rightarrow b^-} f(x)$  exists  $\Leftrightarrow$  for  $\varepsilon > 0 \exists \delta > 0$  s.t.  $a < b - \delta$

$$|f(x_1) - f(x_2)| < \varepsilon \quad \forall x_1, x_2 \text{ s.t.}$$

$$b - \delta < x_1 < x_2 < b$$



— HW (Similar proof). —

Thm: Let  $\int_a^b f$  be an I.I. at  $b$ . Then  $\int_a^b f$  Converges

$\Leftrightarrow$  for  $\varepsilon > 0 \exists \delta > 0$  s.t.  $a < b - \delta$

$$\left| \int_{x_1}^{x_2} f \right| < \varepsilon \quad \forall b - \delta < x_1, x_2 < b.$$

Proof: We know  $\int_a^b f = \lim_{x \rightarrow b^-} \int_a^x f(t) dt$ . (if exists).

$\therefore \int_a^b f$  exists  $\Leftrightarrow$  for  $\varepsilon > 0 \exists \delta > 0$  s.t.  $a < b - \delta$

$$\& \left| \int_a^{x_1} f - \int_a^{x_2} f \right| < \varepsilon \quad \forall b - \delta < x_1 < x_2 < b.$$

$$\Leftrightarrow \left| \int_{x_1}^{x_2} f \right| < \varepsilon$$

— || —

□

$\therefore$  The above is the Cauchy criterion for I.I.  
of Type - I.



Back to Type IIThm: (A.C. test):Suppose  $\varphi \in \mathcal{B}[a, \infty)$   ~~$\mathcal{R}[a, \infty)$~~ , if  $\int_a^\infty f$  is A.C. then $\int_a^\infty \varphi f$  is also A.C. $\therefore$  "Scaling" by a bdd fn. is okay for A.C.Proof:

Note that

$$|(\varphi f)(x)| = |\varphi(x)| |f(x)|$$

$$\leq \left( \sup_{x \in [a, \infty)} |\varphi(x)| \right) \times |f(x)| \quad \forall x \in [a, \infty)$$

$\underbrace{\hspace{1cm}}_{:= M}$

$$\Rightarrow 0 \leq |\varphi(x) f(x)| \leq M |f(x)| \quad \forall x \in [a, \infty).$$

 $\Rightarrow \varphi f \in \mathcal{R}[a, \infty)$  as  $f \in \mathcal{R}[a, \infty)$ .Finally,  $\int_a^\infty |f|$  Converges, by Comparison test $\int_a^\infty |\varphi| |f|$  also Converges. $\varphi f \in \mathcal{R}[a, b]$   
 $\forall b > a.$   
 $\square$ Now we discuss two important integral tests:Non-A.C.Thm: (Abel's test):Let  $\varphi \in \mathcal{B}[a, \infty)$  & suppose  $\varphi$  is monotonic. If $\int_a^\infty f$  Converges, then  $\int_a^\infty \varphi f$  also Converges.2nd: Dirichlet testBUT: We need to prepare the necessary groundwork background!!Scaling by  
bdd monotonic  
fn. is OK!Proof:  
WAIT

Pause for Riemann integration.

Actually, we need the 2nd MVT. But, we can't avoid 1st !!

Thm: (First MVT for integrals):

Let  $f, g \in R[a, b]$  & let  $f$  keeps the same sign over  $[a, b]$ .

Then  $\exists$   ~~$\xi \in [a, b]$~~   $\xi \in [\inf g, \sup g]$  s.t.

$$\int_a^b fg = g \int_a^b f.$$

Curious equality indeed!!

(Also known as weighted MVT!!)

Proof: WLOG, assume that  $f(x) > 0 \quad \forall x \in [a, b]$ .

[OR, Consider  $-f$ ].

We know  $m \leq g(x) \leq M$

Here  $m = \inf g$  over  $[a, b]$ .  
 $M = \sup g$

" $\because f > 0$ , we have:

$$m f(x) \leq g(x) f(x) \leq M f(x) \quad \forall x.$$

" $\because f, g, fg \in R[a, b]$ , it follows that

$$m \int_a^b f \leq \int_a^b fg \leq M \int_a^b f.$$

$$\Rightarrow \exists g \in [m, M] \text{ s.t. } \int_a^b fg = g \int_a^b f.$$

□

# If  $g \in C[a, b]$ , then  $g = g(c)$  for some  $c \in [a, b]$ .

$$\therefore \int_a^b fg = g(c) \int_a^b f$$

Very useful equality!!

# If  $f \equiv 1$  on  $[a, b]$ , then

$$\int_a^b g = g(c)(b-a).$$

$g \in C[a, b]$

$$\text{i.e. } g(c) = \frac{1}{b-a} \int_a^b g.$$

MVT!!  
[We know this.]

eg: Let  $r \in (0, 1)$ . Then

Standard application.

$$\pi/6 \leq \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-rx^2)}} \leq \frac{\pi}{6} \frac{1}{\sqrt{1-r/4}}.$$

difficult to compute.  
So we estimate: practical approach!!

$$\left. \begin{aligned} \text{Set } f(x) &= \frac{1}{\sqrt{1-x^2}} \\ g(x) &= \frac{1}{\sqrt{1-rx^2}} \end{aligned} \right\} x \in [0, 1/2].$$

Clearly,  $f, g \in C[0, 1]$ . Also  $f(x) > 0 \quad \forall x \in [0, 1/2]$ .

$\therefore$  By 1st MVT,  $\exists \xi \in [0, 1/2]$  s.t.

$$\int_0^{1/2} fg = g(\xi) \int_0^{1/2} f.$$

$$= \frac{1}{\sqrt{1-r\xi^2}} \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx.$$

$$= \frac{\pi/6}{\sqrt{1-r\xi^2}} \quad (\text{HW})$$

$$\therefore \xi \in [0, 1/2], \quad \xi^2 \leq \frac{1}{4} \Rightarrow r\xi^2 \leq \frac{r}{4}. \quad (\because r \in (0, 1))$$

$$\Rightarrow \frac{1}{\sqrt{1-r\xi^2}} \leq \frac{1}{\sqrt{1-r/4}}.$$

$$\therefore \int_0^{1/2} fg \leq \frac{\pi/6}{\sqrt{1-r/4}}. \quad [\text{by } \otimes].$$

Finally, since  $r \in (0, 1)$ ,  $\frac{1}{\sqrt{1-x^2}} \leq (fg)(x) \Rightarrow \pi/6 \leq \int_0^{1/2} fg.$

$$\therefore \pi/6 \leq \int_0^{1/2} fg \leq \frac{\pi/6}{\sqrt{1-r/4}}.$$

Abel's lemma: Let  $\{a_j\}_{j=1}^n$  be a <sup>set of</sup> decreasing <sup>+ve</sup> numbers  
 (i.e.  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ ) &  $\{w_j\}_{j=1}^n$  be a set of real nos.

$$\text{Suppose } \alpha \leq \sum_{j=1}^m w_j \leq \beta \quad \forall m=1, \dots, n,$$

for some  $\alpha, \beta \in \mathbb{R}$ .

$$\text{Then } a_1 \alpha \leq \sum_{j=1}^n a_j w_j \leq a_1 \beta.$$

~~Proof~~ [i.e. If  $\alpha \leq \sum_{j=1}^m w_j \leq \beta \quad \forall m=1, \dots, n$ , then  
<sup>Simply</sup>  $\forall$  decreasing  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ , we have:

$$a_1 \alpha \leq \sum_{j=1}^n a_j w_j \leq a_1 \beta.]$$

$m$ -th partial sum.

Proof: Set  $s_m := \sum_{j=1}^m w_j \quad \forall m=1, \dots, n.$

$$\text{We know } \alpha \leq s_m \leq \beta \quad \forall m=1, \dots, n.$$

$$\begin{aligned} \text{Now } \sum_{j=1}^n a_j w_j &= a_1 s_1 + a_2 (s_2 - s_1) + \dots + a_n (s_n - s_{n-1}) \\ &= (a_1 - a_2) s_1 + (a_2 - a_3) s_2 + \dots + \\ &\quad \dots + (a_{n-1} - a_n) s_{n-1} + a_n s_n. \end{aligned}$$

$$\therefore a_j - a_{j+1} \geq 0 \quad \forall j=1, \dots, n-1 \quad \& \quad s_m \leq \beta \quad \forall m=1, \dots, n,$$

by  $\otimes$ , we have  $\sum_{j=1}^n a_j w_j \leq \beta [(a_1 - a_2) + \dots + (a_{n-1} - a_n) + a_n].$   
 $= \beta a_1.$

$\&$  Since  $\alpha \leq s_m \quad \forall m$ , <sup>by  $\otimes$</sup>  we have:

$$\sum_{j=1}^n a_j w_j \geq \alpha [(a_1 - a_2) + \dots + (a_{n-1} - a_n) + a_n].$$

$$= \alpha a_1.$$

$$\therefore \alpha a_1 \leq \sum_{j=1}^n a_j w_j \leq \beta a_1.$$

$\square$

Now we are ready for the 2<sup>nd</sup> MVT. The linear version is due to Weierstrass. First we prove the initial variant:

Thm. (2<sup>nd</sup> MVT: Bonnet's form):

Let  $f, \varphi \in R[a, b]$ , and suppose  $\varphi \geq 0$  & monotonically decreasing on  $[a, b]$ . Then  $\exists \xi \in [a, b]$  s.t.

$$\int_a^b \varphi f = \varphi(\xi) \int_a^b f$$

$\leftarrow f$  is a kind of "weight"  $f(x)$ .

Proof: Let  $P \in \mathcal{P}[a, b]$ . & assume  $P: a = x_0 < x_1 < \dots < x_n = b$ .

Pick  $\xi_j \in I_j \quad \forall j = 2, \dots, n$  &  $\xi_1 := a$ .

$\therefore \{\xi_j\}_{j=1}^n$  is a tag set of  $P$ .

We know:

$$m_j (x_j - x_{j-1}) \leq \int_{x_{j-1}}^{x_j} f \leq M_j (x_j - x_{j-1})$$

Recall:  
 $I_j = [x_{j-1}, x_j]$

$j = 1, \dots, n$ .  
&  $m_j = \inf_{I_j} f$

$M_j = \sup_{I_j} f$

$$\& m_j (x_j - x_{j-1}) \leq f(\xi_j) (x_j - x_{j-1}) \leq M_j (x_j - x_{j-1})$$

$\forall j = 1, \dots, n$ .

By taking partial sums

$$\sum_{j=1}^t m_j |I_j| \leq \int_a^{x_t} f \leq \sum_{j=1}^t M_j |I_j|$$

$$\& \sum_{j=1}^t m_j |I_j| \leq \sum_{j=1}^t f(\xi_j) |I_j| \leq \sum_{j=1}^t M_j |I_j|$$

Combining above pair of inequalities.

$$\left| \int_a^{x_t} f - \sum_{j=1}^t f(\xi_j) |I_j| \right| \leq \sum_{j=1}^t (M_j - m_j) |I_j|$$

$\forall t = 1, \dots, n$ .

$\forall t = 1, \dots, n$ .

$$\Rightarrow \left| \int_a^{x_t} f - \sum_{j=1}^n f(\eta_j) |I_j| \right| \leq \sum_{j=1}^n (M_j - m_j) |I_j|.$$

(89)

Jaydeb Sarkar

$$\Rightarrow \int_a^{x_t} f - \sum_{j=1}^n (M_j - m_j) |I_j| \leq \underbrace{\sum_{j=1}^n f(\eta_j) |I_j|}_{\substack{\text{circled} \\ \forall t=1, \dots, n.}} \leq \int_a^{x_t} f + \sum_{j=1}^n (M_j - m_j) |I_j|$$

(†)

Now we observe that  $x \mapsto \int_a^x f(t) dt$  is a cont. fn. on  $[a, b]$  ( $\because f \in R[a, b]$ ).

In particular:

$$\delta_1 := \min_{x \in [a, b]} \int_a^x f \leq \int_a^y f \leq \delta_2 := \sup_{x \in [a, b]} \int_a^x f. \quad \forall y \in [a, b]$$

$$\therefore (\dagger) \Rightarrow \delta_1 - \sum_{j=1}^n (M_j - m_j) |I_j| \leq \sum_{j=1}^n f(\eta_j) |I_j| \leq \delta_2 + \sum_{j=1}^n (M_j - m_j) |I_j|$$

$$\underbrace{\quad}_{:= \operatorname{osc}_P f = U(f, P) - L(f, P)} \quad \forall t = 1, \dots, n.$$

Set  $a_j := \varphi(\eta_j) \quad j = 1, \dots, n.$

By assumption:  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0.$

Therefore, we are in the setting of Abel's lemma, with:

$$\alpha := \delta_1 - \operatorname{osc}_P f \leq \sum_{j=1}^t \underbrace{f(\eta_j) |I_j|}_{:= w_j} \leq \beta := \delta_2 + \operatorname{osc}_P f. \quad \forall t = 1, \dots, n.$$

$a_1 \geq a_2 \geq \dots \geq a_n.$

By Abel's lemma:

$$a_1 \alpha \leq \sum_{j=1}^n a_j \omega_j \leq a_1 \beta.$$

i.e.  $\varphi(a) \alpha \leq \sum_{j=1}^n \varphi(y_j) f(y_j) |I_j| \leq \varphi(a) \beta.$

$[\because a_1 = \varphi(y_1) = \varphi(a)]$

Note that:  $\text{osc}_P f = \sum_{j=1}^n (M_j - m_j) |I_j|$   
 $= U(f, P) - L(f, P).$

~~$\varphi(a) \left[ S_1 - \sum \right]$~~

The Riemann Sum

$$\therefore \varphi(a) \times \left[ S_1 - (U(f, P) - L(f, P)) \right] \leq R(\varphi f, P) \leq \varphi(a) \times \left[ S_2 + (U(f, P) - L(f, P)) \right]$$

$\forall P \in P[a, b].$

$\therefore S_1, S_2$  are independent of  $P$ , as  $\|P\| \rightarrow 0$ ,

$$\varphi(a) S_1 \leq \int_a^b \varphi f \leq \varphi(a) S_2.$$

~~$\lim_{\|P\| \rightarrow 0} R(\varphi f, P) = \int_a^b \varphi f$~~

$[\because \|P\| \rightarrow 0 \Rightarrow U(f, P) - L(f, P) \rightarrow 0]$

$\& R(\varphi f, P) \rightarrow \int_a^b \varphi f.$

But  $S_1 = \min_{x \in [a, b]} \int_a^x f$  &  $S_2 = \max_{x \in [a, b]} \int_a^x f$ . &  $x \mapsto \int_a^x f$  is cont.

$\Rightarrow \int_a^b \varphi f = \varphi(a) \int_a^b f$  for some  $\eta \in [a, b].$

$\square$

Thm: (2<sup>nd</sup> MVT: Weierstrass' form).

Let  $f, \varphi \in \mathcal{R}[a, b]$  &  $\varphi$  is monotonic on  $[a, b]$ . Then  $\exists \xi \in [a, b]$  s.t.

$$\int_a^b \varphi f = \varphi(a) \int_a^{\xi} f + \varphi(b) \int_{\xi}^b f.$$



Proof: WLOG: assume  $\varphi$  is  $\uparrow$  [otherwise, consider  $-\varphi$ ].

Set  $\tilde{\varphi}(x) := -\varphi(x) + \varphi(b) \quad \forall x \in [a, b]$ .

$\therefore \tilde{\varphi} \in \mathcal{R}[a, b]$ ,  $\tilde{\varphi} \geq 0$  &  $\tilde{\varphi}$  monotonically decreasing ✓

on  $[a, b]$ . By 2<sup>nd</sup> MVT, Bonnet's form,  $\exists \xi \in [a, b]$

s.t. 
$$\int_a^b \tilde{\varphi} f = \tilde{\varphi}(a) \int_a^{\xi} f.$$

$$\Rightarrow -\int_a^b \varphi f + \varphi(b) \int_a^b f = -(\varphi(a) - \varphi(b)) \int_a^{\xi} f. \quad \checkmark$$

$$\begin{aligned} \Rightarrow \int_a^b \varphi f &= \varphi(a) \int_a^{\xi} f + \varphi(b) \left[ \int_a^b f - \int_a^{\xi} f \right] \\ &= \varphi(a) \int_a^{\xi} f + \varphi(b) \int_{\xi}^b f. \end{aligned}$$



Back to Type II improper integration:

We want to prove two tests:



Thm (Abel's test): Let  $\varphi \in B[a, \infty)$  be a monotonic f.  
Let  $\int_a^\infty f$  converges. Then  $\int_a^\infty \varphi f$  also converges.

Proof: We know  $f \in R[a, R] \forall R > a$ . Let  $a < R_1 < R_2$ .

Proof: By 2<sup>nd</sup> MVT (Weierstrass version),  $\exists \xi \in [R_1, R_2]$  s.t.

$$\int_{R_1}^{R_2} \varphi f = \varphi(\xi) \int_{R_1}^{R_2} f \quad \text{--- } (*)$$

Let  $M := \sup_{x \in [a, \infty)} |\varphi(x)|$ , let  $\varepsilon > 0$ .

$\because \int_a^\infty f$  converges,  $\exists R_0 \in \mathbb{R}$  s.t.  $R_0 \in \mathbb{R}$  s.t.

Cauchy criterion / test  $\Rightarrow \left| \int_{B_1}^{B_2} f \right| < \varepsilon / 2M \quad \forall B_1, B_2 \geq R_0 \quad \text{--- } (**)$

Assume  $R_1, R_2 \geq R_0$ . Then  $|\varphi(R_1)|, |\varphi(R_2)| \leq M$ .

hence,  $(*) \Rightarrow (**)$

$$\left| \int_{R_1}^{R_2} \varphi f \right| \leq |\varphi(R_1)| \left| \int_{R_1}^{R_2} f \right| + |\varphi(R_2)| \left| \int_{R_1}^{R_2} f \right|$$
$$\leq M \times \frac{\varepsilon}{2M} + M \times \frac{\varepsilon}{2M} = \varepsilon$$

$$\Rightarrow \left| \int_{R_1}^{R_2} \varphi f \right| < \varepsilon \quad \forall R_1, R_2 \geq R_0$$

$\Rightarrow \int_a^\infty \varphi f$  converges (by Cauchy criterion)

Thm: (Dirichlet Test):

$f \in R[a, R]$   
 $\forall R > a$

Let  $\varphi \in B[a, \infty)$  be a monotonic fn. s.t.  $\lim_{x \rightarrow \infty} \varphi(x) = 0$ .

Suppose  $f \in R[a, \infty)$  s.t.  $x \mapsto \int_a^x f$  is a bdd fn. on  $[a, \infty)$ . Then  $\int_a^\infty \varphi f$  Converges.

Proof: Let  $M := \sup_{x \in [a, \infty)} \left| \int_a^x f \right|$ .

Let  $\varepsilon > 0$ . As  $\lim_{x \rightarrow \infty} \varphi(x) = 0$ ,  $\exists m_0 \in \mathbb{R}$  s.t.

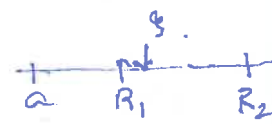
$$|\varphi(x)| < \varepsilon/4M \quad \forall x > m_0.$$

Suppose  $R_1, R_2 > m_0$ .

By 2<sup>nd</sup> MVT (of Weierstrass form),  $\exists \xi$  between  $R_1$  &  $R_2$  s.t.

$$\begin{aligned} \left| \int_{R_1}^{R_2} \varphi f \right| &= \left| \varphi(R_1) \int_{R_1}^{\xi} f + \varphi(R_2) \int_{\xi}^{R_2} f \right| \\ &\leq \underbrace{|\varphi(R_1)|}_{< \varepsilon/4M} \left| \int_{R_1}^{\xi} f \right| + \underbrace{|\varphi(R_2)|}_{< \varepsilon/4M} \left| \int_{\xi}^{R_2} f \right| \\ &< \frac{\varepsilon}{4M} \times \left( \left| \int_{R_1}^{\xi} f \right| + \left| \int_{\xi}^{R_2} f \right| \right) \end{aligned}$$

$\therefore |\varphi(x)| < \frac{\varepsilon}{4M}$   
 $\forall x > m_0$   
 $\& R_1, R_2 > m_0$



Now  $\left| \int_{R_1}^g f \right| = \left| \int_a^g f - \int_a^{R_1} f \right|$

$$\leq \left| \int_a^g f \right| + \left| \int_a^{R_1} f \right|$$

$$\leq 2M$$

$$\left[ \because M = \sup_{x \in [a, \infty)} \left| \int_a^x f \right| \right]$$

$$\frac{1}{4} \left| \int_a^{R_2} f \right| \leq 2M$$

$$\therefore \left| \int_{R_1}^{R_2} \phi f \right| < \frac{\varepsilon}{4M} (2M + 2M) = \varepsilon$$

i.e.  $\left| \int_{R_1}^{R_2} \phi f \right| < \varepsilon \quad \forall R_1, R_2 > m_0$

$$\Rightarrow \int_a^\infty \phi f \text{ Converges.}$$

[by Cauchy test]

□

eg:  $\int_1^\infty \frac{1}{x} \sin x \cdot \log x \cdot dx$

$\int_1^\infty \sin x \, dx$  does not converge

Set  $f(x) = \sin x$ ,  $\phi(x) = \frac{\log x}{x}$

Now  $\int_1^x \sin t \, dt = \cos 1 - \cos x \Rightarrow \left| \int_1^x f \right| \leq 2 \quad \forall x \in [1, \infty)$

$$\Rightarrow \sup_{x \in [1, \infty)} \left| \int_1^x f \right| \leq 2$$

Also,  $\phi(x) \downarrow$  &  $\phi(x) \rightarrow 0$  as  $x \rightarrow \infty$  [Why?]

$\therefore$  By Dirichlet test,  $\int_1^\infty \phi f$  Converges. □

Eg:  $\int_1^{\infty} \frac{\sin x}{x^p} dx$

Set  $f(x) = \sin x$ .

We already know:

~~$\int_1^x \sin t dt$~~

$$\left| \int_1^x f \right| \leq 2 \quad \forall x \in [1, \infty)$$

Also, if  $\varphi(x) := \frac{1}{x^p}$ , then  $\varphi \downarrow$  &  $\varphi(x) \rightarrow 0$  as  $x \rightarrow \infty$

whenever  $p > 0$ .

$\therefore$  By Dirichlet test

$$\int_1^{\infty} \frac{\sin x}{x^p} dx \text{ Converges } \forall p > 0.$$

(\*)

Remark:

#  $\therefore \int_0^1 \frac{\sin x}{x} dx$  Converges, in particular,

$$\int_0^{\infty} \frac{\sin x}{x} dx \text{ Converges.}$$

$$\left[ \because \int_0^{\infty} = \int_0^1 + \int_1^{\infty} \right]$$

#  $\int_1^{\infty} \frac{\sin x}{x^p} dx$  is A.C. whenever  $p > 1$ .

Q: What about A.C?  
-WAIT-

Indeed:  $\left| \frac{\sin x}{x^p} \right| \leq \frac{1}{x^p} \quad \forall x \geq 1$

&  $\int_1^{\infty} \frac{1}{x^p} dx$  Conv. whenever  $p > 1$ .

$\therefore$  By Comparison test:  $\int_1^{\infty} \frac{\sin x}{x^p} dx$  is A.C.

$$\forall p > 1.$$

# In fact, ~~Since~~ <sup>We know</sup>  $\int_1^{\infty} \frac{1}{x^p} dx$  Converges  $\Leftrightarrow p > 1$ ,

~~by limit Comparison Test,~~

Note that  $|\sin x| \geq \sin^2 x$ .

So, if  $0 < p \leq 1$ , then:

$$\left| \frac{\sin x}{x^p} \right| \geq \frac{\sin^2 x}{x^p} = \frac{1 - \cos 2x}{2x^p} \quad (x \geq 1).$$

As in (\*) above,  $\int_1^{\infty} \frac{\cos 2x}{2x^p} dx$  Converges  $\forall p > 0$ .  
*(by Dirichlet test)*

But  $\int_1^{\infty} \frac{1}{2x^p} dx$  diverges  $\forall p \leq 1$ .

$$\Rightarrow \int_1^{\infty} \frac{1 - \cos 2x}{2x^p} dx \text{ diverges } \forall p \leq 1.$$

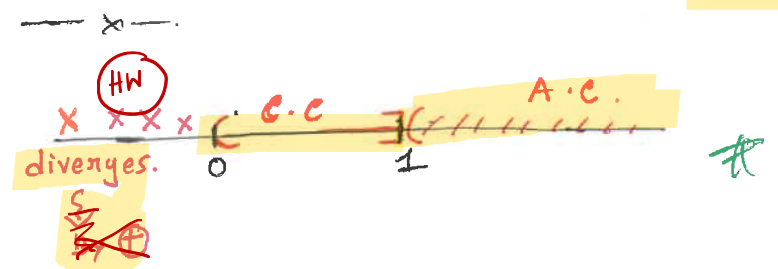
$\therefore$  By Comparison Test:

$$\int_1^{\infty} \left| \frac{\sin x}{x^p} \right| dx \text{ diverges } \forall p \leq 1.$$

$$\therefore \int_1^{\infty} \frac{\sin x}{x^p} dx \text{ is A.C. } \Leftrightarrow p > 1.$$

Then, (\*)  $\Rightarrow \int_1^{\infty} \frac{\sin x}{x^p} dx$  is Conditionally Converges *i.e. Converges but not A.C.*  
for all  $0 < p \leq 1$ .  
 $\int x^p \sin x$  by parts.

So, for  $\int_1^{\infty} \frac{\sin x}{x^p} dx$ ,



We know:  $\int_0^{\infty} \frac{\sin x}{x} dx$  Converges, BUT:

Known as the Dirichlet integral.

Thm:  $\int_0^{\infty} \frac{\sin x}{x} dx$  is Conditionally Converges.

Proof: All we need to prove is that  $\int_0^{\infty} \frac{\sin x}{x} dx$  is NOT A.C.

Fix  $n \in \mathbb{N}$ . Set  $f(x) = \frac{\sin x}{x}$ .

Now  $\int_0^{m\pi} |f| = \sum_{m=1}^n \int_{(m-1)\pi}^{m\pi} |f|$

(See Page-95).

$\forall m = 1, \dots, n$ , we have

$$\int_{(m-1)\pi}^{m\pi} |f| = \int_{(m-1)\pi}^{m\pi} \frac{|\sin x|}{x} dx.$$

$\because x \geq 0$

$$= \int_0^{\pi} \frac{|\sin x|}{(m-1)\pi + x} dx.$$

using  $(x \mapsto (m-1)\pi + x)$ .

$$= \int_0^{\pi} \frac{\sin x}{(m-1)\pi + x} dx.$$

$\because \sin x \geq 0$   
 $\forall x \in [0, \pi]$

Now  $(m-1)\pi + x \leq m\pi \quad \forall x \in [0, \pi]$ .

$$\Rightarrow \frac{1}{(m-1)\pi + x} \geq \frac{1}{m\pi} \quad \forall x \in [0, \pi].$$

$$\Rightarrow \frac{\sin x}{(m-1)\pi + x} \geq \frac{\sin x}{m\pi} \quad \forall x \in [0, \pi].$$

$$\therefore \int_{(m-1)\pi}^{m\pi} |f| \geq \frac{1}{m\pi} \int_0^{\pi} \sin x dx = \frac{1}{m\pi} (\cos 0 - \cos \pi)$$

$$= \frac{2}{m\pi} \quad \forall m = 1, 2, \dots, n.$$

i.e.  $\int_{(m-1)\pi}^{m\pi} |f| \geq \frac{2}{m\pi} \quad \forall m=1, \dots, n.$

$$\therefore \int_0^{n\pi} |f| = \sum_{m=1}^n \int_{(m-1)\pi}^{m\pi} |f|$$

$$\geq \frac{2}{n} \times \sum_{m=1}^n \frac{1}{m} \quad \forall n \in \mathbb{N}.$$

$\therefore \sum_{m=1}^{\infty} \frac{1}{m}$  is divergent, it follows that

$$\lim_{n \rightarrow \infty} \int_0^{n\pi} \left| \frac{\sin x}{x} \right| dx = \infty.$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_0^R \left| \frac{\sin x}{x} \right| dx = \infty.$$

$\lim_{x \rightarrow \infty} f(x)$

Indeed, for  $R \in \mathbb{R}$ ,  $\exists n \in \mathbb{N}$  s.t.

$$n\pi \leq R < (n+1)\pi.$$

$$\Rightarrow \int_0^R |f| \geq \int_0^{(n+1)\pi} |f|$$

$\eta(n)$

$\eta(R)$

$\therefore$  By Comparison test,  $\int_0^R |f| \rightarrow \infty.$

□

HW: Use similar method/tricks to prove that

$$\int_2^{\infty} \frac{\cos x}{\log x} dx \text{ is Conditionally Convergent.}$$

□





On the other hand: for  $\varepsilon > 0$ , a similar calculation yields:

$$\int_{\varepsilon}^1 f = \int_{\varepsilon}^{1/n} (-1)^{n+1} (n+1) dx + \int_{1/n}^{1/(n-1)} (-1)^n n dx + \dots + \int_{1/3}^{1/2} (-1)^2 2 dx + \int_{1/2}^1 1 dx.$$

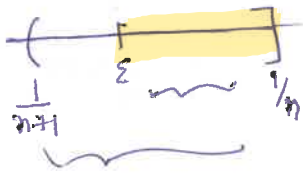
$$= (-1)^{n+1} (n+1) \left( \frac{1}{n} - \varepsilon \right) + (-1)^n \frac{1}{n-1} + \dots + \frac{1}{3} - \frac{1}{2} + 1.$$

$$\Rightarrow \int_{\varepsilon}^1 f - \sum_{m=1}^{n-1} (-1)^{m+1} \frac{1}{m} = (-1)^{n+1} (n+1) \left( \frac{1}{n} - \varepsilon \right).$$

$$\Rightarrow \left| \int_{\varepsilon}^1 f - \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{m} \right| = \left| (n+1) \left( \frac{1}{n} - \varepsilon \right) \right|$$

$$= (n+1) \times \left( \frac{1}{n} - \varepsilon \right)$$

$$< (n+1) \times \left( \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{n}.$$



$$\Rightarrow \left| \int_{\varepsilon}^1 f - \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{m} \right| < \frac{1}{n}.$$

$\therefore$  the alternating series  $\sum_{m=1}^{\infty} (-1)^{m+1} / m$  converges,

Since  $\varepsilon \rightarrow 0^+ \Rightarrow n \rightarrow \infty$ , it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 f = \lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}.$$

$\therefore \int_0^1 f$  is conditionally convergent.  $\square$

Similar idea applies to the following:

Thm: (Cauchy - MacLaurin test) Let  $f \downarrow$ ,  $f(x) > 0$  on  $[1, \infty)$ . Then  $\int_1^\infty f$  &  $\sum_{n=1}^\infty f(n)$  Converge or diverge together.

Proof: Fix  $n \in \mathbb{N}$ . By assumption:

$$f(n) \geq f(x) \geq f(n+1) \quad \forall x \in [n, n+1]$$

$$\Rightarrow f(n) \geq \int_n^{n+1} f \geq f(n+1) \quad \forall n \in \mathbb{N}.$$

~~Let  $s_n$~~  Set  $S_n := \sum_{m=1}^n f(m)$ .  $\forall n \in \mathbb{N}$   
↗ n-th partial sum

$$\therefore \textcircled{*} \Rightarrow S_{n-1} \geq \int_1^n f \geq \sum_{m=1}^{n-1} f(m+1), \quad \forall n \geq 2.$$

$= \sum_{m=2}^n f(m)$

$$\Rightarrow S_{n-1} \geq \int_1^n f \geq \sum_{m=2}^n f(m). \quad \textcircled{**}$$

Observe that:  $\lim_{R \rightarrow \infty} \int_1^R f = \alpha \in \mathbb{R}$ ,  $\iff \lim_{n \rightarrow \infty} \int_1^n f = \alpha$  \*\*\*

$\therefore$  If  $\lim_{n \rightarrow \infty} \int_1^n f = \alpha$  (i.e.  $\int_1^\infty f$  converges), then

$$\textcircled{**} \Rightarrow 0 \leq \sum_{m=2}^n f(m) \leq \int_1^n f \leq \alpha \quad \left[ \because \int_1^n f \uparrow \right]$$

$$\Rightarrow \{S_n\}_{n=1}^\infty \text{ is bdd} \Rightarrow \sum_{n=1}^\infty f(n) \text{ converges.}$$

(102)

If  $\int_1^{\infty} f$  diverges, then ~~then~~ then using  $(**)$ ; i.e.,

$$\int_1^n f \leq S_{n-1}$$

we have that  $\{S_n\}_{n=1}^{\infty}$  unbounded ( $\nearrow$ ).

$$\Rightarrow \sum_{n=1}^{\infty} f(n) \text{ diverges.}$$

$$\therefore \int_1^{\infty} f \text{ Converges (/diverges)} \Rightarrow \sum_{n=1}^{\infty} f(n) \text{ Converges (/diverges).}$$

||y:  $(***) \not\Leftarrow (**)$  implies:

$$\sum_{n=1}^{\infty} f(n) \text{ Converges/div.} \Rightarrow \int_1^{\infty} f \text{ Conv./div.}$$

□

eg: p-series:  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ,  $p > 0$ .

Consider  $\int_1^{\infty} \frac{1}{x^p} dx$ .

We know the above I.T. Conv.  $\forall p > 1$  & div.  $\forall p \leq 1$ .

$\therefore$  The p-series Conv.  $\forall p > 1$  & div.  $\forall 0 < p \leq 1$ .

A known fact !!

Gamma fn.:

Recall the notion of factorial:  $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$   $\forall n \in \mathbb{N}$ .

Q: What about  $x! = ?$  ( $x \in \mathbb{R}_{>0}$ ) !!

The hint is (as we will also see soon):

$$\int_0^{\infty} x^n e^{-x} dx = n! \quad \forall n \geq 1$$

Also, a hw.

We need to generalize this concept!!

Def:  $\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt$   $\forall x > 0$ .

Euler's gamma fn. on  $(0, \infty)$ .

Thm:  $\Gamma(x)$  converges  $\forall x > 0$ .

Proof:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

$$= \underbrace{\int_0^1 t^{x-1} e^{-t} dt}_{:= \gamma_1(x)} + \underbrace{\int_1^{\infty} t^{x-1} e^{-t} dt}_{:= \gamma_2(x)}$$

For  $\gamma_1(x) = \int_0^1 t^{x-1} e^{-t} dt$ , we observe: For any  $x > 0$ ,

$$0 < t^{x-1} e^{-t} < t^{x-1}$$

$$\forall t \in (0, 1]$$

$$[\because e^{-t} < 1]$$

$e \in (0, \infty) \subseteq \mathbb{R}(0, \infty)$   
 $\mathbb{R}[e, \infty) \subseteq \mathbb{R}(0, \infty)$   
 $\forall \epsilon > 0$

Two issues:  $0$  &  $\infty$ .

Type I Type II

$\therefore \int_0^1 t^{x-1} dt = \frac{1}{x}$  , by Comparison test, it follows that  $\gamma_1(x)$  Converges.

For  $\gamma_2(x) = \int_1^\infty t^{x-1} e^{-t} dt$  ,  $x > 0$ ,

We note  $\lim_{t \rightarrow \infty} (t^{x+1} e^{-t}) = 0$  HW.  
(use L'Hospital rule a couple of times)  
[  $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0 \quad \forall p \in \mathbb{R} \]$  .

$\therefore \exists M > 1$  s.t.  $t^{x+1} e^{-t} \leq 1 \quad \forall t \geq M$ .

$\Rightarrow t^{x-1} e^{-t} \leq \frac{1}{t^2} \quad \forall t \geq M$

But  $\int_M^\infty \frac{1}{t^2} dt$  Converges ( $p=2$  case).

$\therefore$  By Comparison test:  $\gamma_2(x) = \int_1^\infty t^{x-1} e^{-t} dt$  Converges  $\forall x > 0$ .

$\int_1^\infty = \int_1^M + \int_M^\infty$

$\Rightarrow \Gamma(x)$  Converges  $\forall x > 0$ . □

Thm:

$\Gamma(x+1) = x \Gamma(x) \quad \forall x > 0$

Expected property.

Proof:  $\forall a < b$ , using by parts:

$$\int_a^b t^{x-1} e^{-t} dt = \left[ -t^{x-1} e^{-t} \right]_a^b + (x-1) \int_a^b t^{x-2} e^{-t} dt$$
  

$$= \left( a^{x-1} e^{-a} - b^{x-1} e^{-b} \right) + (x-1) \int_a^b t^{x-2} e^{-t} dt$$
  
 $\downarrow$   
0 as  $a \rightarrow 0$  &  $b \rightarrow \infty$ .

Proof: Note that  $\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt$ .

Now  $\forall r_1, r_2 > 0$ , we have, by parts, that

$$\begin{aligned} \int_{r_1}^{r_2} t^x e^{-t} dt &= -t^x e^{-t} \Big|_{t=r_1}^{r_2} + x \int_{r_1}^{r_2} t^{x-1} e^{-t} dt \\ &= \left( r_1^x e^{-r_1} - r_2^x e^{-r_2} \right) + x \int_{r_1}^{r_2} t^{x-1} e^{-t} dt \end{aligned}$$

$$\int_0^{\infty} t^x e^{-t} dt = \lim_{r_2 \rightarrow \infty} \left( \lim_{r_1 \rightarrow 0} \int_{r_1}^{r_2} t^x e^{-t} dt \right)$$

as  $r_1 \rightarrow 0$   $\rightarrow 0$  as  $r_2 \rightarrow \infty$ .  
 $\rightarrow 0$  as  $r_1 \rightarrow 0$  &  $r_2 \rightarrow \infty$ .

$$\Rightarrow \int_0^{\infty} t^x e^{-t} dt = x \int_0^{\infty} t^{x-1} e^{-t} dt.$$

$$\Rightarrow \Gamma(x+1) = x \Gamma(x) \quad \forall x > 0. \quad \square$$

#  $\therefore \forall n \in \mathbb{N}$ ,

$$\begin{aligned} \Gamma(n+1) &= n \Gamma(n) \\ &= n(n-1) \Gamma(n-1) \\ &= \dots = \\ &= n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1 \cdot \Gamma(1). \end{aligned}$$

$$\text{Now } \Gamma(1) = \int_0^{\infty} e^{-t} dt = 1.$$

$$\Rightarrow \Gamma(n+1) = n! \quad \forall n \geq 1.$$

$\therefore \Gamma$  is a Cont. analog of factorial  $n!$  !!

$! : \mathbb{N} \rightarrow \mathbb{N}$ .  
 $\Gamma : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$   
 s.t.  $\Gamma|_{\mathbb{N}} = !$

§

# Cauchy principle value :

Recall: if  $f: (-\infty, \infty) \rightarrow \mathbb{R}$  & if  $\exists c \in \mathbb{R}$  s.t.

both I.I's  $\int_{-\infty}^c f$  &  $\int_c^{\infty} f$  exist, then

we say  $\int_{-\infty}^{\infty} f$  converges & write

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^c f + \int_c^{\infty} f$$

In this case, RHS is indep. of the choice of  $c$ .

#  $\therefore \int_{-\infty}^{\infty} f$  converges  $\Leftrightarrow \lim_{R_1, R_2 \rightarrow \infty} \int_{-R_1}^{R_2} f$  exists,

Moreover, in this case:  $\int_{-\infty}^{\infty} f = \lim_{\substack{R_1 \rightarrow \infty \\ R_2 \rightarrow \infty}} \int_{-R_1}^{R_2} f$

That is

~~Indeed:~~

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^c f + \int_c^{\infty} f$$

$$= \lim_{R_1 \rightarrow +\infty} \int_{-R_1}^c f + \lim_{R_2 \rightarrow \infty} \int_c^{R_2} f$$

def

$$= \lim_{\substack{R_1 \rightarrow \infty \\ R_2 \rightarrow \infty}} \int_{-R_1}^{R_2} f$$

0 c m1 m2

= m2

# Therefore,  $\int_{-\infty}^{\infty} f = \lim_{R_1, R_2 \rightarrow \infty} \int_{-R_1}^{R_2} f$  exists & the

limit ~~converg~~ exists independently of how  $R_1$  &  $R_2$  approach  $\infty$ .

$\therefore$  limit of the fn.  $\eta(R_1, R_2) = \int_{-R_1}^{R_2} f$  as  $R_1, R_2 \rightarrow \infty$ .

Clearly, a strong assumption.

& in many cases, this fails to exist.

Instead:

Def: The Cauchy principle value (CPV) of  $\int_{-\infty}^{\infty} f$  is

defined by :

$$\text{CPV} - \int_{-\infty}^{\infty} f = \lim_{R \rightarrow \infty} \int_{-R}^R f \quad (\text{if exists}).$$

Fact: If  $\int_{-\infty}^{\infty} f$   $\checkmark$  exists, then  $\text{CPV} - \int_{-\infty}^{\infty} f$  exists &  $\checkmark$

$\therefore \int_{-R}^R = \int_{-R}^c + \int_c^R$

$\text{CPV} - \int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f$   $\checkmark$

— HW —  
Easy.

#

The Converse is NOT true:



eg: CPV  $\int_{-\infty}^{\infty} f$  exists but  $\int_{-\infty}^{\infty} f$  does not Converge, where

$$f(x) = \frac{x}{1+x^2} \quad x \in \mathbb{R}.$$

Indeed,

$$\begin{aligned} \int_{-R}^R \frac{x}{1+x^2} dx &= \frac{1}{2} \int_{-R}^R \frac{2x}{1+x^2} dx = \frac{1}{2} \int_{1+R^2}^{1+R^2} \frac{1}{t} dt \\ &= \frac{1}{2} \left[ \log t \right]_{1+R^2}^{1+R^2} = 0. \quad \forall R > 0. \end{aligned}$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x}{1+x^2} dx = 0.$$

i.e. CPV  $\int_{-\infty}^{\infty} f = 0.$

However,  $\int_{-\infty}^{\infty} f = \lim_{R \rightarrow \infty} \int_{-R}^0 \frac{x}{1+x^2} dx + \lim_{R \rightarrow \infty} \int_0^R \frac{x}{1+x^2} dx.$

$$= \lim_{R \rightarrow \infty} \frac{1}{2} \int_{1+R^2}^1 \frac{1}{t} dt = -\frac{1}{2} \log$$

$$= \lim_{R \rightarrow \infty} \left\{ -\frac{1}{2} \log(1+R^2) \right\} = -\infty. \quad \checkmark$$

$$= \lim_{R \rightarrow \infty} \frac{1}{2} \int_1^{1+R^2} \frac{1}{t} dt = \infty.$$

$$\Rightarrow \int_{-\infty}^{\infty} f \text{ does not Converge.}$$



# Sequence & Series of functions.

Recall: A seqn.  $\{x_n\} \subseteq \mathbb{R}$  is convergent if  $\exists x \in \mathbb{R}$  s.t.  
for  $\varepsilon > 0 \exists N \in \mathbb{N} - \emptyset$ .

$$|x_n - x| < \varepsilon \quad \forall n \geq N.$$

— (x)



For  $\varepsilon > 0 \exists N \in \mathbb{N} - \emptyset$ .

$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq N.$$

Cauchy  
Criterion.

Goal: Replace  $x_n$  by  $f_n$  ( $: S \rightarrow \mathbb{R}$ ,  $S \subseteq \mathbb{R}$ ) &  
talk about closeness, limit, etc.!!

# Setting: 1)  $S \subseteq \mathbb{R}$ .

2)  $\mathcal{F}(S) = \{f: S \rightarrow \mathbb{R}\}$

Even an algebra.

A vector space  
over  $\mathbb{R}$ .

3)  $\{f_n\}_{n=1}^{\infty}$  or simply  $\{f_n\}$  will  
refer a seqn. of  $f_n$ 's :  $S \rightarrow \mathbb{R}$ .  
i.e.  $\{f_n\} \subseteq \mathcal{F}(S)$ .

Goal (again) : To talk about

$$|f_n - f_m| < \varepsilon$$

$$\forall n, m \geq N$$

A Q:

How to do it?

(2)

2 - ways to do it !!

$$|f - g| = ??$$

Simply Consider  $\{f_n(x)\} \subseteq \mathbb{R}$   
 $\forall x \in S$ . Then ~~int~~ bring back  
 $|\cdot|$  in  $\mathbb{R}$  to the Convergence.

pointwise

modules.

Think a little more:

Introduce a Suitable  
 Concept of distance  
 between 2 functions!!  
 Like "modulus"  
 of functions!!

uniform

def: Let  $\{f_n\} \subseteq \mathcal{F}(S)$  &  $f \in \mathcal{F}(S)$ . We say that  
 $f_n$  Converges to  $f$  pointwise (or  $f_n \xrightarrow{p} f$ ) if

$$\forall x \in S, \quad f_n(x) \longrightarrow f(x).$$

# We also write:  $f_n \longrightarrow f$  pointwise.

$$\# \quad f_n \xrightarrow{p} f \iff \text{For } x \in S \text{ & } \varepsilon > 0 \quad \exists N \in \mathbb{N} \text{ s.t.}$$

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$$

DANGER !!

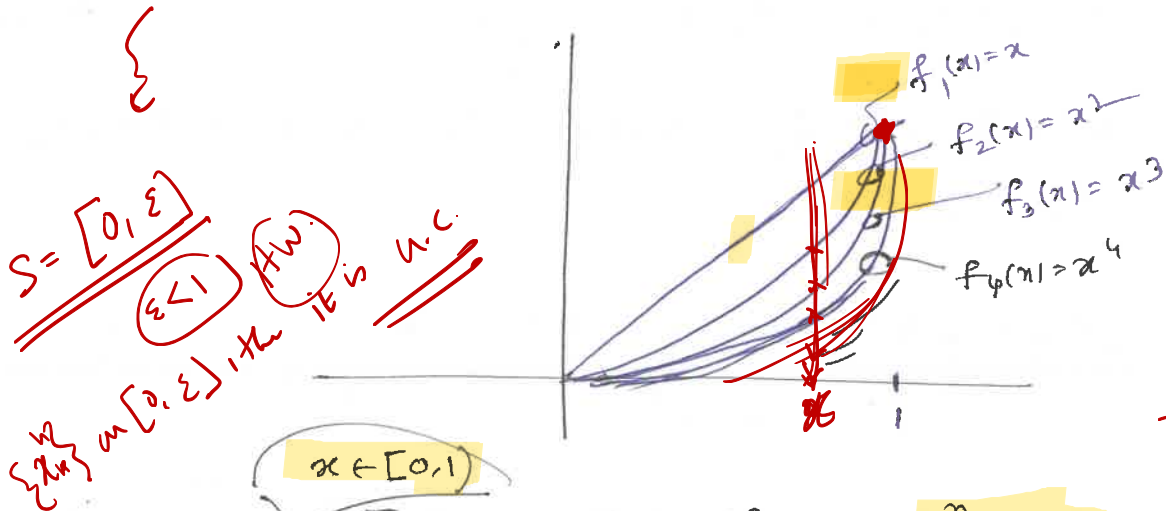
$$N \equiv N(x, \varepsilon).$$

#  $N$  depends on  $\varepsilon$  is acceptable (like  $x_n \rightarrow x$  case),  
 but the dependence on  $x$  is slightly "less desirable!!"

Will be fixed in the  
 "uniform" part.

eg:

1)  $S = [0, 1]$ .  $f_n(x) = x^n \quad \forall x \in [0, 1], n \in \mathbb{N}$ .



For each  $x \in [0, 1)$ , we know that  $x^n \rightarrow 0$ .

For  $x=1$ ,  $x^n \rightarrow 1$ .

$\therefore$  If we define  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases},$$

then  $f_n \xrightarrow{p} f$ .

[Remark:  $\forall n \in \mathbb{N}$ ,  $f_n \in C[0, 1]$ , or even  $f_n \in \mathcal{D}[0, 1]$ . However, the limit (pointwise)  $f_n \notin C[0, 1] !!$

Strange, but reality. We need to fix or identify the trouble!!

(2)  $f_n(x) = \frac{1}{x+n} \quad \forall n \geq 1, x \in [0, \infty).$

$\therefore \forall x \in [0, \infty), f_n(x) \rightarrow 0. \quad [\because \frac{1}{x+n} \leq \frac{1}{n} \rightarrow 0]$

$\Rightarrow f_n \xrightarrow{p} f$  where  $f(x) = 0 \quad \forall x \in [0, \infty).$

i.e.  $f_n \xrightarrow{p} 0.$

[But something more: For  $\varepsilon > 0$ , we have:

$$|f_n(x) - 0| = \frac{1}{x+n} \leq \frac{1}{n},$$

$\therefore |f_n(x) - 0| < \varepsilon \quad \forall n > \frac{1}{\varepsilon}.$

$n$  does not depend on  $x$ !!

Def: Let  $\{f_n\} \subseteq \mathcal{F}(S)$  &  $f \in \mathcal{F}(S)$ . We say that  $f_n$  converges to  $f$  uniformly (or write  $f_n \xrightarrow{u} f$ , or  $f_n \rightarrow f$  unif.) if  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

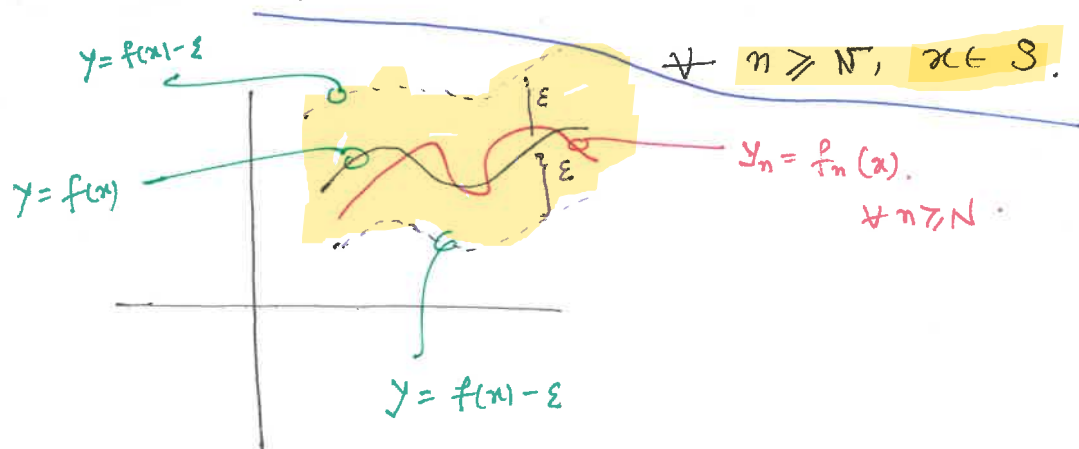
$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N \text{ & } x \in S.$$

————— (\*)

eg:  $f_n \xrightarrow{u} 0$ , where  $f_n(x) = \frac{1}{x+n} \quad \forall x \in [0, \infty) \text{ & } n \in \mathbb{N}.$

Remark:

(\*)  $\Leftrightarrow f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$



$$[x_n \rightarrow x : |x_n - x| < \frac{1}{n} \forall n \geq N]$$

5

Jaydeb Sarkar

Def: For  $S \subseteq \mathbb{R}$ ,  $\mathcal{B}(S) := \{f: S \rightarrow \mathbb{R} \text{ bdd.}\}$

Set of all bdd fns on  $S$ .

A vector space over  $\mathbb{R}$ .

[Here  $(\alpha f + g)(x) = \alpha f(x) + g(x)$ ,  
 $\forall \alpha \in \mathbb{R}, f, g \in \mathcal{B}(S)$ ]

Def:  $\forall f \in \mathcal{B}(S)$ , define  $\|f\|$  (read: norm of  $f$ )

by  $\|f\| = \sup_{x \in S} |f(x)|$ .

Also known as "the sup norm".

Example:  $f(x) = x$ ,  $g(x) = 1-x$ ,  $S = [0,1]$   
 $\|f\| = 1 = \|g\|$   
 $\|f+g\| = \frac{1}{4}$ .  
 $\Delta$ -inequality.

Remark:

(1)  $\|\cdot\| : \mathcal{B}(S) \rightarrow \mathbb{R}_{\geq 0}$ .

(2)  $\|f\| = 0 \iff f \equiv 0$ .

(3)  $\|f+g\| \leq \|f\| + \|g\|$ .

(4)  $\|\alpha f\| = |\alpha| \|f\| \quad \forall \alpha \in \mathbb{R}$ .

(5)  $\|fg\| \leq \|f\| \|g\|$

Submultiplicative.

Looks like  $|\cdot|$  on  $\mathbb{R}$ !!

Remark: Indeed,  $\|\cdot\|$  on  $\mathcal{B}(S)$  plays the role of  $|\cdot|$  on  $\mathbb{R}$ !!

Def:  $\forall f, g \in \mathcal{B}(S)$ , define the distance between  $f$  &  $g$

as:

$d(f, g) = \|f - g\|$ .

Also known as metric on  $\mathcal{B}(S)$ .

Remark:  $\therefore d: \mathcal{B}(S) \times \mathcal{B}(S) \rightarrow \mathbb{R}_{\geq 0}$ . And:

(1)  $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$ .

(2)  $d(f, g) \leq d(f, h) + d(h, g)$ .

(3)  $d(f, g) = 0 \iff f = g$ .

$\forall f, g \in \mathcal{B}(S)$ .



(6)

$= \{g: S \rightarrow \mathbb{R}\}$

# Let  $\{f_n\} \subseteq \mathcal{F}(S)$  &  $f \in \mathcal{F}(S)$ . Recall:  $f_n \xrightarrow{u} f$  <sup>on S</sup> if  $f_n \xrightarrow{u} f$   $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in S, n \geq N.$$

$$\Leftrightarrow (\because f_n - f \in \mathcal{B}(S) \quad \forall n \geq N)$$

$$\|f_n - f\| < \varepsilon \quad \forall n \geq N.$$

$N \geq n$   
 $|x_n - x| < \varepsilon$

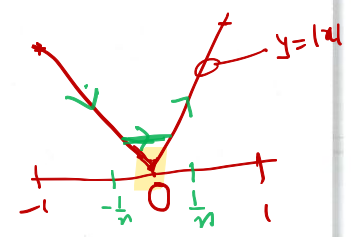
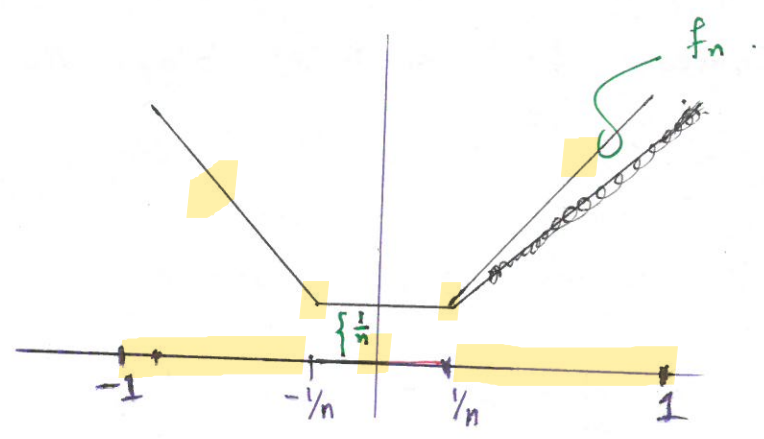
Now this "looks" like modulus.

Ensures that  $f_n - f \in \mathcal{B}(S) \quad \forall n \geq N$ .

eg:

Let  $S = [-1, 1]$ . Define  $f_n \in \mathcal{F}(S)$  by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq \frac{1}{n} \\ |x| & \text{if } \frac{1}{n} < |x| \leq 1 \end{cases}$$



Then  $\lim_{n \rightarrow \infty} f_n(x) = |x| \quad \forall x \in S$ .

i.e.  $f_n \xrightarrow{p} f$  on  $S$ , where

$$f(x) = |x| \quad \forall x \in S.$$

\* fixed  $x \in [-1, 1]$

$$|f_n(x) - f(x)| = \begin{cases} \left| \frac{1}{n} - |x| \right| & \text{if } |x| \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < |x| \leq 1 \end{cases} \Rightarrow f_n(x) \rightarrow f(x) \quad \forall x$$

In fact:  $\|f_n - f\| = \sup_{x \in [-1, 1]} |f_n(x) - f(x)|$

$$= \frac{1}{n} \quad \forall n. \quad \leftarrow \because \text{the max occurs at } x=0.$$

$\therefore$  For  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$$\|f_n - f\| < \varepsilon \quad \forall n \geq N.$$

$$\Rightarrow f_n \xrightarrow{u} f \text{ on } [-1, 1].$$

□

Remark:  $f_n \xrightarrow{u} f \Rightarrow f_n \xrightarrow{p} f$ . (i.e. unif  $\Rightarrow$  point.)

Proof: Suppose that  $f_n \xrightarrow{u} f$  uniformly on  $S$ .

Let  $\varepsilon > 0$ . Then  $\exists N \in \mathbb{N}$  s.t.

$$\|f_n - f\| < \varepsilon \quad \forall n \geq N.$$

$$\text{i.e. } \sup_x |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

$$\text{i.e. } |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, x \in S.$$

$$\Rightarrow f_n(x) \rightarrow f(x) \quad \forall x \in S.$$

$$\therefore f_n \rightarrow f \text{ pointwise.} \quad \square$$

#  $\therefore$  For u.c. it is perhaps a good idea to compute the pointwise limit (if exists) first !!

[If pointwise limit fails to exist, then u.c. also fails.]

Remark #  $\Leftarrow$  of the above is NOT true! i.e. pointwise  $\not\Rightarrow$  unif.

eg:  $f_n(x) = x^n$  on  $[0, 1]$ .  $\leftarrow \{f_n\}$  is NOT u.c. (HW).

# Now, we develop some useful tools for convergence !!

The above examples suggest a pattern!

Although computation of pointwise limit may be diff. occasionally.



(8)

Thm: (Cauchy's criterion): Let  $\{f_n\} \subseteq \mathcal{F}(S)$ .

~~$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in S.$$~~

~~$$[i.e. f_n \xrightarrow{p} f]$$~~

Then  $\{f_n\}$  is u.c.

(This is a must for u.c.)

Then  $f_n \xrightarrow{u} f \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ s.t.}$

$$\|f_m - f_n\| < \varepsilon \quad \forall m, n \geq N.$$

No need to worry about limit  $f_n \rightarrow f$ .

(Also ensures that  $f_m - f_n \in \mathcal{B}(S) \quad \forall m, n$ ).

Proof: " $\Rightarrow$ " Let  $f_n \xrightarrow{u} f$ , and let  $\varepsilon > 0$ .

$$\therefore \exists N \in \mathbb{N} \text{ s.t.} \quad \|f_n - f\| < \varepsilon/2 \quad \forall n \geq N.$$

$\therefore \forall m, n \geq N$ , we have:

$$\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\|$$

$$\begin{aligned} &\leq \|f_n - f\| + \|f - f_m\| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

$\Delta$ -ineq.

Standard proof like for  $\{x_n\} \subseteq \mathbb{R}$

" $\Leftarrow$ " Let  $\varepsilon > 0$ .

$$\therefore \exists N \in \mathbb{N} \text{ s.t.}$$

$$\|f_n - f_m\| < \varepsilon/2 \quad \forall m, n \geq N.$$

$$= \sup_{x \in S} |f_n(x) - f_m(x)|$$

i.e.  ~~$\{f_n(x)\}_{n \geq 1}$  is Cauchy  $\forall x \in S$ .~~

$$i.e. \quad |f_n(x) - f_m(x)| < \varepsilon/2 \quad \forall m, n \geq N \quad \forall x \in S. \quad (*)$$

$\Rightarrow \{f_n(x)\}$  is Cauchy  $\forall x \in S. \Rightarrow \lim_{n \rightarrow \infty} f_n(x) := f(x)$  exists  $\forall x \in S$ .

~~$$i.e. f_n \xrightarrow{p} f$$~~

$$\text{Also } (*) \Rightarrow f_n(x) - \varepsilon/2 < f_m(x) < f_n(x) + \varepsilon/2 \quad \forall m, n \geq N, x \in S.$$

But  $\lim_{m \rightarrow \infty} f_m(x) = f(x) \quad \forall x \in S.$

⑨  
Jaydeb Sarkar  
 $f(x) := \lim_{n \rightarrow \infty} f_n(x).$

$\therefore$  Taking limit as  $m \rightarrow \infty$ , we have.

$$f_n(x) - \varepsilon/2 \leq f(x) \leq f_n(x) + \varepsilon/2 \quad \forall n \geq N \quad \forall x \in S.$$

$$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon$$

$$\forall n \geq N \quad \forall x \in S.$$

$$\Rightarrow \|f_n - f\| < \varepsilon \quad \forall n \geq N.$$

i.e.  $f_n \xrightarrow{u} f$  on  $S.$

$= \{f_n : n \in \mathbb{N}\}.$

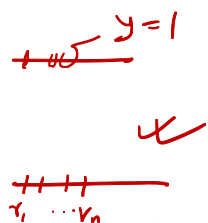
$f : \mathbb{N} \rightarrow \mathbb{Q} \cap [0, 1]$

# Cauchy Criterion is often useful.

Eg: Let  $\{r_n\}_{n=1}^{\infty}$  be an enumeration of rationals  $\mathbb{Q} \cap [0, 1].$

For each  $n \in \mathbb{N}$ , define  $f_n : [0, 1] \rightarrow \mathbb{R}$  by

$$f_n(x) = \begin{cases} 0 & \text{if } x = r_1, \dots, r_n \\ 1 & \text{if } x \neq r_1, \dots, r_n. \end{cases}$$



Then  $\{f_n\}$  is not u.c. on  $[0, 1].$

Indeed, choose  $\varepsilon = \frac{1}{2}.$

Observe, for each  $m \in \mathbb{N}$ ,  $f_m(r_m) = 0$  &  $f_m(r_{m+1}) = 1$

we have  $f_n(r_{n+1}) = 1$  &  $f_{n+1}(r_{n+1}) = 0.$

$$\Rightarrow |f_n(r_{n+1}) - f_{n+1}(r_{n+1})| = 1. \quad \forall n.$$

$$\therefore \nexists N \in \mathbb{N} \text{ s.t. } |f_m(x) - f_n(x)| < \varepsilon = \frac{1}{2}.$$

$$\forall m, n \geq N.$$

$\therefore$  By Cauchy Criterion,  $\{f_n\}$  is NOT u.c.

[Q:  $\{f_n\}$  Converges pointwise?]

Thm: (M-test): Let  $\{f_n\} \subseteq \mathcal{F}(S)$  & suppose  $f_n \xrightarrow{p} f$ .

Set  $M_n := \sup_{x \in S} |f_n(x) - f(x)|$ .

Then  $f_n \xrightarrow{u} f \text{ on } S \iff M_n \longrightarrow 0$ .

Proof: " $\Rightarrow$ " Let  $f_n \xrightarrow{u} f$ , & let  $\varepsilon > 0$ .

$\therefore \exists N \in \mathbb{N} \cdot \forall n \geq N, \|f_n - f\| < \varepsilon$

$\stackrel{\text{By def.}}{=} M_n$

i.e.  $M_n < \varepsilon \quad \forall n \geq N$ .

$\Rightarrow M_n \longrightarrow 0$ .

" $\Leftarrow$ " Let  $\varepsilon > 0$ . Then  $\exists N \in \mathbb{N} \cdot \forall n \geq N, M_n < \varepsilon$

$\Rightarrow \|f_n - f\| < \varepsilon \quad \forall n \geq N$ .

$\Rightarrow f_n \xrightarrow{u} f$ .

$\Rightarrow f_n \xrightarrow{u} f$ .

eg:

$\forall n \in \mathbb{N}, x \in [0, 1]$ , define

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

Note that:  $\frac{nx}{1+n^2x^2} < 1 \quad \forall n$  & fixed  $x \in [0, 1]$ .

$\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{i.e.} \quad f_n \xrightarrow{p} f$ ,

where  $f(x) \equiv 0$ .

$\therefore M_n = \|f_n - f\| = \sup_{x \in [0, 1]} \left[ \frac{nx}{1+n^2x^2} \right] \quad \forall n$ .

$\frac{nx}{1+n^2x^2} \rightarrow 0$  as  $n \rightarrow \infty$  as  $x$  is fixed.

$f_n(x) < 1 \quad \forall n$   
 $\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0$

fix  $x \in [0, 1]$ ,  $\lim_{n \rightarrow \infty} f(x) = 0$

But,  $x=0 \Rightarrow f_n(x) = 0 \quad \forall n$ .

$$\& x > 0 \Rightarrow \frac{\frac{1}{nx} + nx}{2} \geq \sqrt{\frac{1}{nx} \times nx} = 1.$$

$$\Rightarrow \frac{1}{2} \times \frac{1+n^2x^2}{nx} \geq 1.$$

$$\Rightarrow \frac{nx}{1+n^2x^2} \leq \frac{1}{2}.$$

And, "=" occurs at  $x = \frac{1}{n}$ .

$$\therefore M_n = \frac{1}{2} \quad \forall n \geq 1.$$

$$\Rightarrow M_n \not\rightarrow 0.$$

$\therefore$  By M-test,  $\{f_n\}$  is not unif. convergent on  $[0,1]$ .

□

eg:  $\forall n \in \mathbb{N}$ , define  $f_n(x) = x^n(1-x)$ .  $x \in [0,1]$ .

$$\left. \begin{aligned} f_n(x) &= \begin{cases} \therefore f_n(1) = 0 = f_n(0) \quad \forall n. \\ \& \text{for } 0 < x < 1, f_n(x) \rightarrow 0. \end{cases} \end{aligned} \right\}$$

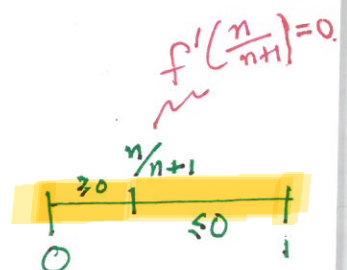
$$\therefore f_n \xrightarrow{p} 0 \quad \text{the zero fn.}$$

$$\therefore M_n = \sup_{x \in [0,1]} [x^n(1-x)].$$

$$\text{But } f'_n(x) = x^{n-1} [n - (n+1)x].$$

$$\therefore \underline{f'_n(x) \geq 0} \quad \forall x \in [0, \frac{n}{n+1}]$$

$$\& \underline{f'_n(x) \leq 0} \quad \forall x \in [\frac{n}{n+1}, 1].$$





(12)

$\therefore f_n(0) = f_n(1) = 0$ , it follows that  $f_n$  has a max.

at  $x = \frac{n}{n+1}$ .

$$\text{Now } f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \times \frac{1}{n+1} < \frac{1}{n+1}.$$

$$\text{i.e. } M_n < \frac{1}{n+1}.$$

$$\Rightarrow M_n \rightarrow 0.$$

$\therefore$  By M-test,  $\{f_n\}$  is u.c.  $(\& f_n \xrightarrow{u} 0)$ .  $\square$

~~Eg~~ Consider  $S = \mathbb{R}$ .

Remark: Suppose  $|f_n(x) - f(x)| \leq M_n \quad \forall x \in S, \quad n \geq 1.$

If  $M_n \rightarrow 0$ , then  $f_n \xrightarrow{u} f$  on  $S$ .

all  $n$  but finitely many.

Proof: Follows from M-test.

Eg: Let  $r > 0$ .  $f_n(x) := e^{-nx} \quad \forall x \in [r, \infty).$

Now  $f_n \downarrow$  on  $\mathbb{R} \quad \forall n$ .

$$\Rightarrow 0 < f_n(x) = e^{-nx} \leq e^{-nr} \quad \forall n \geq 1, x \geq r.$$

But  $e^{-nr} \rightarrow 0$  as  $n \rightarrow \infty$ .

$\Rightarrow f_n \rightarrow 0$  uniformly on  $[r, \infty)$ .  $\square$

Compatibility :- (Examples) :

eg: Recall  $\{f_n\} \subseteq C[0,1]$ , where  $f_n(x) = x^n$ ,  $x \in [0,1]$ ,  $n \in \mathbb{N}$ .

Then  $f_n \xrightarrow{p} f$ , where  $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$ .

Clearly,  $f \notin C[0,1]$ .

$\therefore$  pointwise convergence & continuity are not compatible.

[Q: If  $\{f_n\}$  are cont. &  $f_n \xrightarrow{u} f$  on  $S$

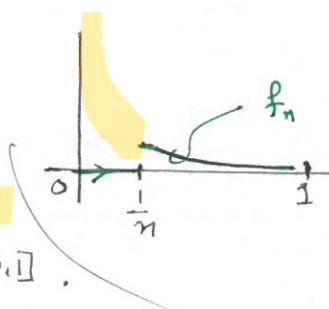
$\Rightarrow$   $f$  is cont? ] Also, if  $f_n$  is cont. at

#  $x_0$ , does  $\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n = \lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f_n (= \lim_{x \rightarrow x_0} f)$ ?

eg:  $f_n(x) = \begin{cases} 0 & x \in [0, 1/n] \\ 1/x & x \in [0,1] \setminus [0, 1/n] \end{cases}$

$\therefore \{f_n\} \subseteq B[0,1]$ .

However,  $\lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & \text{if } x=0 \\ 1/x & \text{if } x \in (0,1] \end{cases}$



$\Rightarrow \lim_{n \rightarrow \infty} f_n(x) := f(x) \quad \forall x \in [0,1]$

is NOT bdd.

$\therefore$  pointwise convergence does not preserve boundedness!!

[Q: What about u.c.?

If  $f_n \xrightarrow{u} f$   
 $\Rightarrow f_n \xrightarrow{p} f$

$$\mathbb{Q} \cap [0,1] = \{r_n\}_{n=1}^{\infty}$$

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eg: Consider an enumeration  $\{r_n\}_{n=1}^{\infty}$  of rationals  $\mathbb{Q} \cap [0,1]$ .

Define 
$$f_n(x) = \begin{cases} 0 & \text{if } x = r_1, \dots, r_n \\ 1 & \text{if } x \in [0,1] \setminus \{r_1, \dots, r_n\} \end{cases}$$

$$\therefore f_n \in \mathcal{R}[0,1] \quad \forall n.$$

[ $\because f_n$  is discontin. at finitely many points.]

Now for  $m \in \mathbb{N}$ , we know:  $f_n(r_m) = 0 \quad \forall n \geq m.$

$$\Rightarrow f_n(r_m) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \forall m \in \mathbb{N}$$

$$\therefore \forall x \in \{r_n\}_{n=1}^{\infty}, \quad \lim_{n \rightarrow \infty} f_n(x) = 0.$$

Next, let  $x \in [0,1] \setminus \mathbb{Q}$ .

$$\therefore f_n(x) = 1 \quad \forall n \Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 1.$$

$$\therefore f_n \xrightarrow{p} f \text{ on } [0,1],$$

where 
$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

But we know that  $f \notin \mathcal{R}[0,1]$ .

$$\therefore \lim_{n \rightarrow \infty} f_n(x) \notin \mathcal{R}[0,1]$$

$\therefore \mathcal{R}[a,b]$  is not closed under pointwise convergence!!

Q: What if  $f_n \rightarrow f$  unif.?

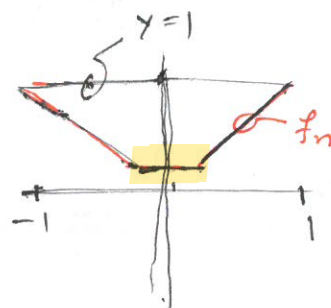
Even if  $\lim f_n = f \in \mathcal{R}[a,b]$

$$\Rightarrow \int \lim f_n = \lim \int f_n$$

eg: Recall that  $f_n \rightarrow f$  uniformly on  $[-1, 1]$ , where

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq \frac{1}{n} \\ |x| & \text{if } \frac{1}{n} < |x| \leq 1 \end{cases}$$

$$f(x) = |x|, \quad x \in [-1, 1].$$



Note that  $f_n$  is diff. at 0  $\forall n$ .

However,  $f$  is NOT diff. at 0.

# Here the situation is even worse: as  $f_n \rightarrow f$  unif. on  $[-1, 1]$ .

⇓  
[ u.c. is not compatible with diff!! ]

All the examples yield negative feeling about the following compatibility issue:

Fact Suppose  $\{f_n\} \subseteq \mathcal{F}(S)$ ,  $f \in \mathcal{F}(S)$ . Suppose  $f_n \rightarrow f$  pointwise on  $S$ .

# Let  $f_n$  is cont. on  $S \forall n$ .  $\stackrel{?}{\Rightarrow} f$  is cont. on  $S$ ? **NO!**

# Let  $f_n \in \mathcal{B}(S) \forall n$ .  $\stackrel{?}{\Rightarrow} f \in \mathcal{B}(S)$ ? **NO!**

# Let  $f_n \in \mathcal{R}[a, b] \forall n$ .  $\stackrel{?}{\Rightarrow} f \in \mathcal{R}[a, b]$ ? **NO!**

If so, then must it be true that

$$\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b \lim_{n \rightarrow \infty} f_n \quad \left( = \int_a^b f \right) \quad ? \quad \leftarrow ??$$

(pending)

# Let  $f_n$  is diff. at  $x \in S$ ,  $\forall n$ .  $\stackrel{?}{\Rightarrow} f'$  exists at  $x$ ? **NO!**

If so, then must it be true that

$$\lim_{n \rightarrow \infty} f'_n(x) = f'(x) \quad ? \quad \leftarrow ??$$

(pending)



# Suppose  $\lim_{n \rightarrow \infty} f_n$  exists  $\forall n$ .  $\stackrel{?}{\Rightarrow} \lim_{n \rightarrow \infty} f$  exist?

NO!

If so, is it true that

$$\lim_{n \rightarrow \infty} \left( \lim_{n \rightarrow \infty} f_n \right) = \lim_{n \rightarrow \infty} f$$

?

~~Pending~~

NO

eg:  
See Page-13

$$f_n(x) = \begin{cases} 0 & 0 < x \leq 1/n \\ 1/n & 1/n < x \leq 1 \end{cases} \quad f_n: (0,1] \rightarrow \mathbb{R}.$$

$\Rightarrow \lim_{n \rightarrow \infty} f_n$  exists  $\forall n$ . But  $\lim_{n \rightarrow \infty} \left( \lim_{n \rightarrow \infty} f_n \right)$  DNE !!

eg: Consider  $f_n(x) = x^n$ ,  $x \in (0,1)$ .

$\frac{1}{x}$

We know,  $\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x$ , where  $f \equiv 0$ .

i.e.  $f_n \rightarrow 0$  pointwise on  $(0,1)$ .

Now,  $\lim_{n \rightarrow \infty} f_n(x) = 1 \quad \forall x$ .

But  $\lim_{n \rightarrow \infty} f(x) = 0$ .

$$\therefore \lim_{n \rightarrow \infty} \lim_{n \rightarrow \infty} f_n(x) \neq \lim_{n \rightarrow \infty} \lim_{n \rightarrow \infty} f_n(x)$$

- x -

All in all:

Pointwise Convergence is a natural concept but with a number of disadvantages !!

AND, Indeed, one would like to capture all the above properties of convergence !!

In the following, we prove that with uniform convergence,  
all problems disappear.

BUT, NOT <sup>for</sup> with differentiability!!  $\longleftrightarrow$  We will work this out too!!

Thm: Let  $x_0 \in S$  &  $f_n \xrightarrow{u} f$  on  $S \setminus \{x_0\}$ . If  $\lim_{x \rightarrow x_0} f_n$  exists  $\forall n$ ,  
 then  $\lim_{x \rightarrow x_0} f$  also exists. In this case,  

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n = \lim_{x \rightarrow x_0} f.$$
  
 [i.e.  $\lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n = \lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f_n$ .]  $\therefore$  Interchange of limits.

u.c. & limit

Proof: Let  $\varepsilon > 0$ . Since  $f_n \xrightarrow{u} f$ , by Cauchy criterion,  
 $\exists N \in \mathbb{N}$  s.t.

$$\|f_n - f_m\| < \varepsilon/2 \quad \forall n, m \geq N. \quad \text{ON } \underline{S \setminus \{x_0\}}.$$

$\forall n \in \mathbb{N}$ , set  $\underline{a_n} := \lim_{x \rightarrow x_0} f_n$ .

Now,  $a_n - a_m = \lim_{x \rightarrow x_0} [f_n(x) - f_m(x)]$   $\forall n, m \geq 1$ .

$$\Rightarrow |a_n - a_m| = \lim_{x \rightarrow x_0} |f_n(x) - f_m(x)| \quad \left[ \begin{array}{l} \because |a-b| \leq |a-b| \\ \leq |a-b| \end{array} \right]$$

$$\leq \frac{\varepsilon}{2} \quad [\text{by } (*)] \quad \forall m, n \geq N$$

$$\Rightarrow |a_n - a_m| \leq \varepsilon/2 \quad \forall m, n \geq N.$$

$\Rightarrow \{a_n\}$  is Cauchy.

$\therefore \exists a \in \mathbb{R}$  s.t.  $\underline{a} := \lim_{n \rightarrow \infty} a_n$ .

$$\therefore a = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n(x).$$

(18)

Again,  $f_n \xrightarrow{u} f$  on  $S \setminus \{x_0\}$  gives:  $\exists n_0 \in \mathbb{N}$  s.t.

$$\|f_n - f\| < \varepsilon/3 \quad \forall n \geq n_0, \text{ on } S \setminus \{x_0\}. \quad \textcircled{i}$$

Also,  $f_n a_n \rightarrow a$ ,  $\exists \tilde{n}_0 \in \mathbb{N}$  s.t.

$$|a_n - a| < \varepsilon/3 \quad \forall n \geq \tilde{n}_0. \quad \textcircled{ii}$$

Set  $\hat{n} := \max\{n_0, \tilde{n}_0\}$ .

Focus is on  $\hat{n}$  now!!

$\therefore \lim_{x \rightarrow x_0} f_{\hat{n}} = a_{\hat{n}}$ ,  $\exists \delta > 0$  s.t.

$$|f_{\hat{n}}(x) - a_{\hat{n}}| < \varepsilon/3 \quad \forall x \in S \setminus \{x_0\} \text{ s.t. } |x - x_0| < \delta. \quad \textcircled{iii}$$

$\therefore$  for each  $x \in S \setminus \{x_0\}$  s.t.  $|x - x_0| < \delta$ , we have:

$$|f(x) - a| \leq |f(x) - f_{\hat{n}}(x)| + |f_{\hat{n}}(x) - a_{\hat{n}}| + |a_{\hat{n}} - a|.$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad (\text{by (i) - (iii)})$$

$$= \varepsilon.$$

$$\therefore |f(x) - a| < \varepsilon \quad \forall x \in S \setminus \{x_0\} \text{ s.t. } |x - x_0| < \delta.$$

$$\Rightarrow \lim_{x \rightarrow x_0} f = a.$$

$$\text{i.e. } \lim_{x \rightarrow x_0} f = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n.$$

□

Typical  $\varepsilon/3$ -argument.

unif. Cont. & limit

Thm: (Continuity) Let  $f_n \xrightarrow{u} f$  on  $S$ . Let  $x_0 \in S$  & let each  $f_n$  is continuous at  $x_0$ . Then  $f$  is also cont. at  $x_0$ .

Proof: We know  $\lim_{n \rightarrow \infty} f_n = f(x_0) \quad \forall n. \quad [\because f_n \text{ is cont. at } x_0].$

$$\text{Also, } f_n \xrightarrow{u} f \Rightarrow \lim_{n \rightarrow \infty} f_n(x_0) = f(x_0).$$

$$\therefore f(x_0) = \lim_{n \rightarrow \infty} f_n(x_0) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n$$

$$= \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n \quad [\text{by previous thm}]$$

$$= \lim_{n \rightarrow \infty} f$$

$\Rightarrow$   $f$  is cont. at  $x_0$ .  $\square$

Thm: (Bounded  $f_n$ 's). Let  $\{f_n\} \subseteq \mathcal{B}(S)$  &  $f_n \xrightarrow{u} f$  on  $S$ .

Then  $f \in \mathcal{B}(S)$ .

$[\therefore \mathcal{B}(S) \text{ is closed under uniform limits.}]$

Proof:  $\because f_n \xrightarrow{u} f$  on  $S$ , for  $\varepsilon = 1$ ,  $\exists N \in \mathbb{N}$  s.t.

$$\|f_n - f\| < 1 \quad \forall n \geq N.$$

Then,  $\forall x \in S$ , we have:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$$

$$< 1 + \|f_N\|.$$

$$\Rightarrow \|f\| < 1 + \|f_N\| \Rightarrow \underline{f \in \mathcal{B}(S)}. \quad \square$$

u.c.  
&  
b.b.d  
↓



Thm: (Riemann integration) Let  $\{f_n\} \subset \mathcal{R}[a, b]$  s.t.  $f_n \xrightarrow{u} f$  on  $[a, b]$ . Then  $f \in \mathcal{R}[a, b]$  s.t.  $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$ .  
 [i.e.  $\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b \lim_{n \rightarrow \infty} f_n$ ] Interchange of Limits.

u.c. integ.

Proof: Let  $\varepsilon > 0$ . Since  $f_n \xrightarrow{u} f$  on  $[a, b]$ ,  $\exists N \in \mathbb{N}$  s.t.

$$\|f_n - f\| < \frac{\varepsilon}{b-a} \quad \forall n \geq N. \quad \text{--- (i)}$$

Focus on  $N$ .

In particular:  $\|f_N - f\| < \frac{\varepsilon}{b-a}$ . i.e.  $\sup_{x \in [a, b]} |f_N(x) - f(x)| < \frac{\varepsilon}{b-a}$ .

i.e.  $f(x) - \frac{\varepsilon}{b-a} < f_N(x) < f(x) + \frac{\varepsilon}{b-a}$ . ii

$\therefore f_N \in \mathcal{R}[a, b]$ ,  $\exists P \in \mathcal{P}[a, b]$  s.t.

$$U(f_N, P) - L(f_N, P) < \varepsilon. \quad \text{--- (iii)}$$

Now (ii)  $\Rightarrow f(x) < f_N(x) + \frac{\varepsilon}{b-a} \quad \forall x \in [a, b]$

$$\Rightarrow U(f, P) < U(f_N, P) + \varepsilon$$

Similarly (ii)  $\Rightarrow f(x) > f_N(x) - \frac{\varepsilon}{b-a} \quad \forall x \in [a, b]$

$$\Rightarrow L(f, P) > L(f_N, P) - \varepsilon$$

So, (iii)  $\Rightarrow U(f_N, P) - L(f_N, P) < \varepsilon$

$$\Rightarrow (U(f, P) - \varepsilon) - (L(f, P) + \varepsilon) < \varepsilon$$

$$\Rightarrow U(f, P) - L(f, P) < 3\varepsilon$$

$$\Rightarrow \underline{f \in \mathcal{R}[a, b]}$$

Finally, we prove that  $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$ .

$\forall x \in [a, b]$ , we have

$$\left| \int_a^x f - \int_a^x f_n \right| = \left| \int_a^x (f_n - f) \right| \leq \int_a^x |f_n - f|$$

$\therefore$  By (i),  $\forall n \geq N$ , we have:

$$\left| \int_a^x f - \int_a^x f_n \right| \leq \frac{\varepsilon}{b-a} \times \int_a^x 1 \, dx = \frac{\varepsilon}{b-a} \times (x-a) \leq \varepsilon \quad \forall x \in [a, b].$$

$\therefore \left\{ \int_a^x f_n \right\}$  converges unif. to  $\int_a^x f$  on  $[a, b]$ .

In particular:  $\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f.$

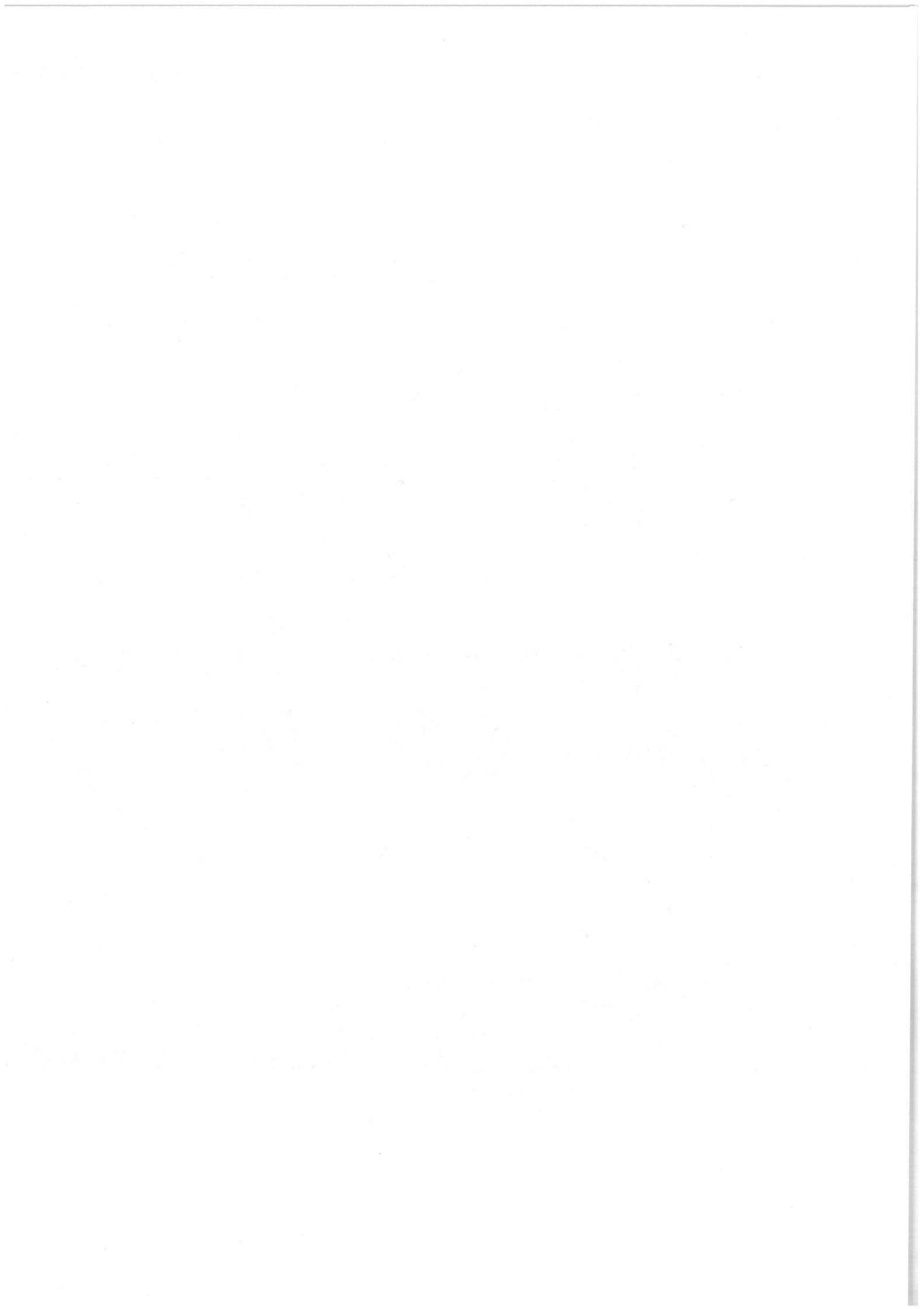
□



In fact: We proved that:

$$\lim_{n \rightarrow \infty} \int_a^x f_n = \int_a^x \lim_{n \rightarrow \infty} f_n \quad \forall x \in [a, b].$$

□



Finally, we prove that  $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$ .

$\forall x \in [a, b]$ , we have

$$\left| \int_a^x f - \int_a^x f_n \right| = \left| \int_a^x (f_n - f) \right| \leq \int_a^x |f_n - f|.$$

$\therefore$  By (i),  $\forall n \geq N$ , we have:

$$\begin{aligned} \left| \int_a^x f - \int_a^x f_n \right| &\leq \frac{\varepsilon}{b-a} \times \int_a^x 1 \, dx = \frac{\varepsilon}{b-a} \times (x-a) \\ &\leq \varepsilon \quad \forall x \in [a, b]. \end{aligned}$$

$\therefore \left\{ \int_a^x f_n \right\}$  converges unif. to  $\int_a^x f$  on  $[a, b]$ .

In particular:  $\lim_{n \rightarrow \infty} \int_a^b f_n = \int_a^b f$ .

□



In fact: we proved that:

$$\lim_{n \rightarrow \infty} \int_a^x f_n = \int_a^x \lim_{n \rightarrow \infty} f_n \quad \forall x \in [a, b].$$

□

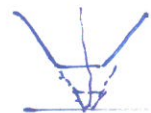


u.c. & differentiability:

Recall:  $f_n \rightarrow f$  &  $f'_n(x_0)$  exists  $\forall n \not\Rightarrow f'(x_0)$  exists.

Another extreme example:

eg:



eg: Suppose  $f_n(x) = \frac{x}{1+nx^2}$ ,  $x \in \mathbb{R}$ ,  $n \geq 1$ .

$\therefore \{f_n\} \subseteq C(\mathbb{R})$ ,

$$|f_n(x)| = \left| \frac{x}{1+nx^2} \right|$$

$$\leq \frac{|x|}{2\sqrt{n}|x|}$$

$$= \frac{1}{2\sqrt{n}}$$

$$\forall x \neq 0.$$

dec.

$$\begin{aligned} & \because 1+nx^2 \geq 2\sqrt{n}|x| \\ & \forall x \in \mathbb{R} \\ & \Rightarrow \frac{1}{1+nx^2} \leq \frac{1}{2} \frac{1}{\sqrt{n}|x|} \\ & \forall x \in \mathbb{R}, n \geq 1, x \neq 0. \end{aligned}$$

$$\therefore |f_n(x)| \leq \frac{1}{2\sqrt{n}} \quad \forall n \geq 1, x \in \mathbb{R}.$$

$\uparrow (\because f_n(0) = 0 \forall n)$ .

$$\Rightarrow f_n \xrightarrow{u} f, \text{ where } f \equiv 0 \text{ on } \mathbb{R}.$$

M-test.

[Note that  $f_n(0) = 0 \forall n \geq 1$ ]

$$\text{Now: } f'_n(x) = \frac{-x \times 2nx + (1+nx^2)}{(1+nx^2)^2} = \frac{-n x^2 + 1}{(1+nx^2)^2}$$

$$= \frac{1-nx^2}{(1+nx^2)^2}$$

$\forall x \neq 0$

$$\therefore \frac{1-n}{(1+n)^2} = \frac{1/n-1}{(1/n^2+1/n)^2} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have  $f'_n(x) \rightarrow F(x) \forall x \in \mathbb{R}$ ,

$$\text{where } F(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

[Note:  $f'_n(0) = 1 \forall n$ ]

In particular,  $f_n'(0) = 1 \quad \forall n$

$$\Rightarrow \lim_{n \rightarrow \infty} f_n'(0) = 1 \neq f'(0) = 0.$$

i.e. <sup>although</sup> ~~even~~  $\{f_n\} \xrightarrow{u} f$  &  $\{f_n\}$  &  $\{f\}$  diff. at  $x_0$ , still

$$\frac{d}{dx} \left( \lim_{n \rightarrow \infty} f_n \right) \neq \lim_{n \rightarrow \infty} \left( \frac{d}{dx} f_n \right) . !!$$

Failure of interchange of limits.

Still, we have some news:

Thm: Let  $\{f_n\}$  be a seqn of diff.  $f_n$  on  $[a, b]$  s.t.

(i)  $\exists x_0 \in [a, b]$  s.t.  $\{f_n(x_0)\}$  converges.

(ii)  $\{f_n'\}$  converges uniformly on  $[a, b]$ .

(iii) Then  $\{f_n\}$  converges uniformly on  $[a, b]$  to a diff.  $f_n \rightarrow f$

$$f'(x) = \lim_{n \rightarrow \infty} f_n'(x) \quad \forall x \in [a, b].$$

Proof: Let  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$  s.t.

$$\left. \begin{aligned} \|f_n' - f_m'\| &< \frac{\epsilon}{2(b-a)} \\ &\& |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \end{aligned} \right\} \quad \forall m, n \geq N.$$

Fix  $m, n \geq N$ ,

& fix  $x \in [a, b] \setminus \{x_0\}$ . By MVT ( $\because f_n - f_m$  is diff.)

$\exists \xi$  between  $x$  &  $x_0$  s.t.  $\leftarrow$  (applied to  $f_n - f_m$ )

$$(f_n - f_m)(x) - (f_n - f_m)(x_0) = (f_n - f_m)'(\xi) * (x - x_0).$$

$$\Rightarrow f_n(x) - f_m(x) = (f_n(x_0) - f_m(x_0)) + (f_n'(\xi) - f_m'(\xi))(x - x_0).$$

$$\Rightarrow |f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |x - x_0| |f'_n(\xi) - f'_m(\xi)|$$

$\Delta$ -ineq.

$$\stackrel{\text{by } (*)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} \times |x - x_0|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \left[ \because |x - x_0| \leq b - a \right]$$

$$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \geq N \text{ \& } x \in [a, b].$$

(xx)

For  $x = x_0$  it is true anyway.

$$\text{i.e. } \|f_n - f_m\| < \varepsilon \quad \forall n, m \geq N.$$

$$\Rightarrow \{f_n\} \text{ is u.c. on } [a, b].$$

$$\text{Let } \lim_{n \rightarrow \infty} f_n = f \text{ on } [a, b].$$

Claim:  $f$  is diff. on  $[a, b]$  &  $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$   
 $\forall x \in [a, b].$

Fix  $\tilde{x} \in [a, b]$ . Set  $\xrightarrow{\text{Next page}}$

$$\tilde{f}_n(x) := \begin{cases} \frac{f_n(x) - f_n(\tilde{x})}{x - \tilde{x}} & \forall x \neq \tilde{x} \\ f'_n(\tilde{x}) & \text{if } x = \tilde{x} \end{cases}$$

$$\tilde{f}(x) = \begin{cases} \frac{f(x) - f(\tilde{x})}{x - \tilde{x}} & \forall x \neq \tilde{x} \\ f' & \end{cases}$$

Fix  $\tilde{x} \in [a, b]$ .

Set  $F_n(x) := \frac{f_n(x) - f_n(\tilde{x})}{x - \tilde{x}}$

$\neq F(x) := \frac{f(x) - f(\tilde{x})}{x - \tilde{x}}$

$\forall n \neq x \neq \tilde{x}$

$\because f_n$  is diff. on  $[a, b] \forall n$ , it follows that

$$\lim_{x \rightarrow \tilde{x}} F_n(x) = f'_n(\tilde{x}) \quad \forall n$$



Also, for each  $x \in [a, b] \setminus \{\tilde{x}\}$ , by MVT,  $\exists \xi$  between  $x$  &  $\tilde{x}$

s.t.  $(f_n - f_m)(x) - (f_n - f_m)(\tilde{x}) = (x - \tilde{x}) \times (f_n - f_m)'(\xi)$

LHS =  $(f_n(x) - f_n(\tilde{x})) - (f_m(x) - f_m(\tilde{x}))$

$\Rightarrow (F_n(x) - F_m(x)) = f'_n(\xi) - f'_m(\xi)$

(Recall:  $\|f'_n - f'_m\| < \varepsilon / 2(b-a)$ ).

$\therefore (*)$  implies,  $|F_n(x) - F_m(x)| < \frac{\varepsilon}{2(b-a)} \quad \forall n, m \geq N$   
 $\forall x \neq \tilde{x}$

$\Rightarrow \|F_n - F_m\| < \frac{\varepsilon}{2(b-a)} \quad \forall n, m \geq N$   
on  $[a, b] \setminus \{\tilde{x}\}$

$\Rightarrow \{F_n\}$  is u.c. on  $[a, b] \setminus \{\tilde{x}\}$ .

interchanging limits.

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow \tilde{x}} F_n(x) = \lim_{x \rightarrow \tilde{x}} \lim_{n \rightarrow \infty} F_n(x)$$

$\downarrow$   $f'_n(\tilde{x})$   $\downarrow$

But  $f_n \xrightarrow{u} f \Rightarrow F_n(x) \rightarrow \frac{f(x) - f(\tilde{x})}{x - \tilde{x}} = F(x)$   
as  $n \rightarrow \infty$

i.e.  $\lim_{n \rightarrow \infty} F_n(x) = F(x) \quad \forall x \neq \tilde{x}$



$$\therefore \lim_{n \rightarrow \infty} \lim_{x \rightarrow \tilde{x}} F_n(x) = \lim_{x \rightarrow \tilde{x}} F(x).$$

$$\therefore (\star) \Rightarrow \lim_{n \rightarrow \infty} f'_n(\tilde{x}) = \lim_{x \rightarrow \tilde{x}} F(x). \quad \text{--- } \textcircled{+}$$

$\therefore F(x) = \frac{f(x) - f(\tilde{x})}{x - \tilde{x}}$  & since the LHS <sup>at  $\textcircled{+}$</sup>  ~~limit~~ exists,

it follows that  $\lim_{x \rightarrow \tilde{x}} F(x) = f'(\tilde{x})$  exists.

$$\therefore f'(\tilde{x}) = \lim_{n \rightarrow \infty} f'_n(\tilde{x}). \quad \square$$

Another exotic example:

$\{f_n\}$  be a diff. fn. &  $f_n \rightarrow f$  unif.

Suppose  $f$  is also diff. Still,  $f'_n \not\rightarrow f'$   
in general!!

Eg:  $f_n(x) = \frac{\sin nx}{n} \quad x \in [0, 1], \quad n \geq 1.$

↑  
Just see the previous ex.

$\because \left| \frac{\sin nx}{n} \right| \leq \frac{1}{n}, \quad \forall n,$  it follows  
that  $f_n \rightarrow f$  unif. on  $[0, 1]$ , where

$$f \equiv 0.$$

$\therefore f' \equiv 0$  on  $[0, 1]$ . Also,  $f'_n(x) = \cos nx$   
 $\forall n, \text{ & } x \in [0, 1].$

But  $f'_n(x)$  does not converge for  $x \in (0, 1]$ .

$\therefore f'_n \not\rightarrow f'.$  (even point

↑  
 $\lim_{n \rightarrow \infty} f'_n(x)$  DNE.

# Series of Functions.

Def. Let  $\{f_n\} \in \mathcal{F}(S)$ . The formal sum of functions

$$f_1 + f_2 + \dots := \sum_{n=1}^{\infty} f_n$$

is called a series of f\_n's.

#  $\forall x \in S$ , by  $\sum f_n(x)$ , we understand the formal sum  $f_1(x) + f_2(x) + \dots$

# Given  $\sum_{n=1}^{\infty} f_n$ ,  $\forall n \in \mathbb{N}$ , define the n-th partial sum of the series as:

$$S_n(x) = \sum_{k=1}^n f_k(x). \quad (\forall x \in S). \quad n \in \mathbb{N}.$$

↑ (Finite sum: So good to go.)

Def: Let  $f: S \rightarrow \mathbb{R}$  &  $\{f_n\}_{n=1}^{\infty} \in \mathcal{F}(S)$ . We say that

$f$  is the pointwise limit of  $\sum f_n$  if  $S_n \rightarrow f$  pointwise on  $S$ .

$\sum f_n = f$ , pointwise.  ~~$S_n \rightarrow f$  on  $S$~~  In this case, we write

If  $S_n \xrightarrow{u} f$  on  $S$ , then we say that  $\sum f_n = f$  uniformly on  $S$ .

eg:  $S = [0, 1)$ .  $f_n(x) = x^n \quad \forall n, x \in [0, 1)$ .

$$\therefore S_n(x) = \sum_{m=0}^{n-1} x^m \Rightarrow S_n(x) = \frac{1-x^n}{1-x} \quad \forall x \in [0, 1)$$

$$\therefore \lim_{n \rightarrow \infty} S_n(x) = \frac{1}{1-x} \quad \forall x \in [0, 1)$$

i.e.  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  pointwise on  $[0, 1)$ .

However,  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  is not uniform on  $[0,1)$ .

Indeed:  $\frac{1}{1-x} \notin \mathcal{B}[0,1)$ . However:

$$S_n \in \mathcal{B}[0,1) \quad \& \quad \lim_{n \rightarrow \infty} S_n(x) = \frac{1}{1-x}.$$

□

Thm: (Cauchy Criterion) Let  $\{f_n\} \subseteq \mathcal{F}(S)$ . Then  $\sum f_n$  converges uniformly on  $S \iff$  for  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$$\left\| \sum_{k=m+1}^n f_k \right\| < \varepsilon \quad \forall n > m \geq N.$$

Proof:  $\because S_n = \sum_{k=1}^n f_k$ , it follows that

$$S_n - S_m = \sum_{k=m+1}^n f_k \quad \forall n > m.$$

Then  $\sum f_n$  is u.c.  $\iff \{S_n\}$  is u.c.  $\iff$  Cauchy Criterion for seq. of f.n.s. □

Cor: If  $\sum f_n$  converges uniformly on  $S$ , then  $\|f_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof:  $\because f_n = S_{n+1} - S_n \quad \forall n$ , this follows from the above theorem. □

$$f_n(x) = \frac{1}{n} \quad \forall x \in \mathbb{R}.$$

eg: Consider  $0 < \varepsilon < 1$  &  $\sum_{n=0}^{\infty} x^n$ . Let  $n > m, x \in [-\varepsilon, \varepsilon]$ .

$$\text{Now } |S_n(x) - S_m(x)| = \frac{1}{|1-x|} \times |(1-x^n) - (1-x^m)|$$

$$= \frac{|x^n - x^m|}{|1-x|} \leq \frac{2|x|^m}{|1-x|} \quad \left( \because n > m \text{ \& } |x| < 1 \right)$$



$$\therefore |S_n(x) - S_m(x)| \leq \frac{2|x|^m}{1-|x|} \leq \frac{2|x|^m}{1-|x|} \quad n > m$$

$$\begin{aligned} |x| < \varepsilon &\Rightarrow -|x| > -\varepsilon \\ \Rightarrow 1-|x| &> 1-\varepsilon \\ \Rightarrow \frac{1}{1-|x|} &< \frac{1}{1-\varepsilon} \end{aligned}$$

$$\leq \frac{2\varepsilon^m}{1-\varepsilon}$$

$$(\because x \in [-\varepsilon, \varepsilon] \text{ for } 0 < \varepsilon < 1)$$

$$\text{i.e. } |S_n(x) - S_m(x)| \leq 2 \times \frac{\varepsilon^m}{1-\varepsilon} \quad \forall x \in [-\varepsilon, \varepsilon] \text{ and } n > m$$

$$\Leftrightarrow \left\| \sum_{k=m+1}^n f_k \right\| \leq 2 \frac{\varepsilon^m}{1-\varepsilon} \quad \forall n > m$$

$0 < \varepsilon < 1$

by Cauchy  
Criterion

$$\begin{aligned} &\because \varepsilon^m \rightarrow 0 \text{ as } m \rightarrow \infty, \text{ for } \tilde{\varepsilon} > 0 \exists N \in \mathbb{N} \\ &\text{s.t. } 2 \frac{\varepsilon^m}{1-\varepsilon} < \tilde{\varepsilon} \quad \forall m \geq N \\ &\therefore \forall n > m \geq N \\ &\left\| \sum_{k=m+1}^n f_k \right\| < \tilde{\varepsilon} \end{aligned}$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n \text{ is u.c. on } [-\varepsilon, \varepsilon] \quad \forall 0 < \varepsilon < 1$$

"  $\therefore \sum_{n=0}^{\infty} x^n$  is u.c. on all compact subsets of  $(-1, 1)$ . "

# Of course:  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall x \in [-\varepsilon, \varepsilon]$

# There are other ways to prove the above conclusion

If  $\sum f_n$  is u.c. on  $S$ , then it is u.c. on  $\tilde{S}$ ,  $\forall \tilde{S} \subseteq S$

Thm: Let  $\{f_n\} \subseteq \mathcal{F}(S)$ ,  $\|f_n\| \leq M_n \quad \forall n \geq 1$  &

Suppose  $\sum_{n=1}^{\infty} M_n < \infty$ . Then  $\sum f_n$  converges uniformly

& absolutely on  $S$

$$\|f_n\| = \sup_{x \in S} |f_n(x)|$$

M-test



[Def:  $\sum f_n$  is absolutely convergent if  $\sum |f_n(x)|$  converges  $\forall x$ .]

Proof:  $\because \|f_n\| < M_n \Rightarrow |f_n(x)| < M_n \forall x \text{ \& } n$ ,

by comparison test,  $\sum f_n$  is absolutely convergent.

Now,  $\forall n > m$ , we have

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n f_k \right\| \leq \sum_{k=m+1}^n \|f_k\| \leq \sum_{k=m+1}^n M_k.$$

$\because \sum_{n=1}^{\infty} M_n < \infty$ , by Cauchy criterion,  $\sum f_n$  is u.c.  $\square$

#  
eg: ①  $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p} \quad p > 1, x \in \mathbb{R}.$

Here  $f_n(x) := \frac{\sin nx}{n^p} \quad \forall n, x \in \mathbb{R}.$

$$\therefore \|f_n\| \leq \frac{1}{n^p}.$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$  converges ( $p > 1$ ), by Weierstrass M-test,

$\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$  converges absolutely & uniformly on  $\mathbb{R}.$

② Set  $f_n(x) = \frac{x}{n+n^2x^2} \quad (x \in \mathbb{R}). \quad \therefore f_n(0) = 0 \forall n.$

For  $x \neq 0$ :  $|f_n(x)| = \frac{|x|}{n+n^2x^2} = \frac{1}{\frac{n}{|x|} + n^2|x|} \leq \frac{1}{2n^{3/2}}$

$$\Rightarrow \|f_n\| \leq \frac{1}{2n^{3/2}}.$$

$\therefore$  By Weierstrass M-test,  $\sum_{n=1}^{\infty} \frac{x}{n+n^2x^2}$  is absolutely & u.c. on  $\mathbb{R}.$   $\square$

③ u.c but NOT absolutely Convergent:

( Not so easy example. But we also have almost trivial one: wait )

Consider  $\sum f_n$  on  $\mathbb{R}$ , with

$$f_n(x) = \frac{(-1)^{n+1}}{n+x^2} \quad \forall x \in \mathbb{R}.$$

i.e.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+x^2}$  on  $\mathbb{R}$ .

useful technique

Now for each fixed  $x \in \mathbb{R}$ ,  $|f_n(x)| = \frac{1}{n+x^2}$ .

But  $\sum_{n=1}^{\infty} \frac{1}{n+x^2}$  is divergent. [Note: For  $x \in \mathbb{R}$  fixed, given  $n \in \mathbb{N} \exists m \in \mathbb{N}$  s.t.  $n+x^2 < n+m$

$$\Rightarrow \frac{1}{n+x^2} > \frac{1}{n+m}]$$

$\Rightarrow \sum f_n$  is not absolutely convergent  $\forall x \in \mathbb{R}$ .

Now we prove that  $\sum f_n$  is u.c.

$\forall n \in \mathbb{N}$ , observe that:

$$S_{2n}(x) = \left( \frac{1}{1+x^2} - \frac{1}{2+x^2} \right) + \left( \frac{1}{3+x^2} - \frac{1}{4+x^2} \right) + \dots + \left( \frac{1}{2n-1+x^2} - \frac{1}{2n+x^2} \right) \quad \forall x \in \mathbb{R}.$$

$\therefore$  all terms are  $> 0$ , it follows that  $S_{2n}(x) \uparrow$ .

Also,  $\sum f_n(x)$  is an alternating series & easy to See it converges  $\forall x \in \mathbb{R}$ . Why?

Set  $f(x) := \sum f_n(x)$   $\forall x \in \mathbb{R}$ .

$$\sum f_n(x) = \lim_{n \rightarrow \infty} S_n(x)$$

$\therefore f(x) - S_{2n}(x) \geq 0 \quad \forall x$ .

$$\text{Also, } f(x) - S_{2n}(x) = \frac{1}{2n+1+x^2} - \left( +ve \text{ no.} \right)$$

$f(x) - S_{2n+1}(x) < 0$

$$< \frac{1}{2n+1+x^2} < \frac{1}{2n+1}$$

$\forall x \in \mathbb{R}$ .



i.e.  $\underline{f(x) - S_{2n}(x) < \frac{1}{2n} \quad \forall x \in \mathbb{R}, n \in \mathbb{N}}$

||y

~~S\_{2n}~~  $S_{2n+1}(x) - f(x) > 0$

~~S\_{2n+1}(x) - f(x) < \frac{1}{2n}~~

$f_n(x) = \frac{(-1)^{n+1}}{n+x^2}$

$S_{2n+1}(x) - f(x) = \frac{S_{2n}(x) - f(x)}{S_{2n}(x) - f(x)} + \frac{f_{2n+1}(x)}{>0}$

$\therefore \forall n \in \mathbb{N} \ \& \ x \in \mathbb{R},$

$$\left. \begin{aligned} 0 < f(x) - S_{2n}(x) < \frac{1}{2n} \\ \& \ 0 < S_{2n+1}(x) - f(x) < \frac{1}{2n} \end{aligned} \right\} \Rightarrow \|S_n - f\| < \frac{1}{2n} \quad \forall n$$

i.e.  $\underline{|S_n(x) - f(x)| < \frac{1}{2n} \quad \forall n \ \& \ x \in \mathbb{R}}$

$\Rightarrow \underline{S_n \xrightarrow{u} f \text{ on } \mathbb{R}}$

$\therefore \sum_{n=1}^{\infty} f_n$  is u.c. on  $\mathbb{R}$

# Even Simpler:  $f_n(x) = \frac{(-1)^{n+1}}{n} \quad \forall n, x$ . Then  $\sum f_n$  is u.c. but NOT A.C.!!

Thm: Suppose  $\sum f_n = f$  uniformly on  $S \setminus \{x_0\}$  for some  $x_0 \in S$ .

if  $\lim_{x \rightarrow x_0} f_n$  exists  $\forall n \in \mathbb{N}$ , then

Limit-Series thm.

$\sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} f_n$  exist ~~exist~~ Converges &

$\lim_{x \rightarrow x_0} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} f_n$  Interchange of limits.

Proof:

For  $\varepsilon > 0 \quad \exists N \in \mathbb{N}$  s.t.  $\| \sum_{k=m+1}^n f_k \| < \varepsilon/2$  i.e.

$\left| \sum_{k=m+1}^n f_k(x) \right| < \varepsilon/2 \quad \forall \ n > m \geq N \ \& \ x \in S \setminus \{x_0\}$

$\therefore \lim_{x \rightarrow x_0} \sum_{k=1}^n f_k$  exists  $\forall n$ , & as the above sum is finite, it follows that

$$\left| \sum_{k=m+1}^n \lim_{x \rightarrow x_0} f_k \right| < \varepsilon \quad \forall \quad n > m \geq N.$$

$$\Rightarrow \alpha := \sum_{n=1}^{\infty} \lim_{x \rightarrow x_0} f_n \text{ Converges.}$$

You can start

the proof from

right  
HERE!

Finally, since  $\sum f_n = f$  unif. on  $S \setminus \{x_0\}$ ,

$$S_n \rightarrow f \text{ uniformly on } S \setminus \{x_0\}$$

$$\therefore S_n = \sum_{k=1}^n f_k \quad \& \quad \lim_{x \rightarrow x_0} f_n \text{ exists } \forall n,$$

$$\text{the limit} \quad \lim_{x \rightarrow x_0} S_n = \sum_{k=1}^n \lim_{x \rightarrow x_0} f_k$$

$\therefore$  By "limit - u.c. thm", it follows that

$$S_n = f_n$$

$$\left[ \lim_{x \rightarrow x_0} S_n \right] \xrightarrow{n \rightarrow \infty} \lim_{x \rightarrow x_0} f \quad \text{as } n \rightarrow \infty.$$

$$\text{i.e.} \quad \lim_{n \rightarrow \infty} \left( \lim_{x \rightarrow x_0} \sum_{k=1}^n f_k \right) = \lim_{x \rightarrow x_0} f$$

$$\text{i.e.} \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \lim_{x \rightarrow x_0} f_k = \lim_{x \rightarrow x_0} \sum_{k=1}^{\infty} f_k$$

□

By using partial sums & corresponding results in u.c. we have the following:

$\{f_n\}$  on  $S \setminus \{x_0\}$ .  
Let  $f_n \rightarrow f$  unif on  $S \setminus \{x_0\}$ . If  $\lim_{x \rightarrow x_0} f_n$  exists then  $\lim_{x \rightarrow x_0} f$  exists.  
we have  $\lim_{x \rightarrow x_0} f(x)$  exists  $\forall n$ .  
 $= \lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f_n$



# Let  $\sum f_n = f$  unif. on  $S$ . If  $f_n$  is bdd  $\forall n$ , then  $f$  is also bdd.

# Let  $\sum f_n = f$  unif. on  $[a, b]$ ,  $f_n \in R[a, b] \forall n$ .  
Then  $f \in R[a, b]$  &  $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$ .

# If  $f_n \in C(S), \forall n$ , &  $\sum f_n$  Converges unif. then  
 $\sum f_n \in C(S)$ .

#  $\| \cdot \|_1$  derivatives.

— HW —

Similar proof.



~~Thm: (Dini's thm on u.c) Let  $S \subseteq \mathbb{R}$  be compact,  $\{f_n\} \subset C(S)$   
&  $f_n \rightarrow f \in C(S)$  pointwise. If  $\{f_n\}$  is monotonically  
decreasing (i.e.  $\{f_n(x)\} \downarrow \uparrow \forall x \in S$ ), then  
 $f_n \rightarrow f$  uniformly on  $S$ .~~

~~Proof: If possible, let  $f_n \not\rightarrow f$  unif. on  $S$ .~~

~~$\therefore \exists \varepsilon > 0$  s.t.  $\|f_n - f\| > \varepsilon$  for infinitely many  $n \in \mathbb{N}$ .~~

~~$$\sup_{x \in S} \{ |f_n(x) - f(x)| \}$$~~

~~$$\sup_{x \in S} \{ f_n(x) - f(x) \}$$~~

~~$$[\because f_n(x) \downarrow f(x) \forall x \in S]$$~~

Thm: (Dini's theorem on u.c.)

Let  $S \subseteq \mathbb{R}$  be compact,  $\{f_n\} \subseteq C(S)$  & let  $f_n \rightarrow f \in C(S)$  pointwise. If  $\{f_n\}$  is monotonic (i.e.  $\{f_n(x)\} \downarrow$  or  $\uparrow \forall x \in S$ ) then  $f_n \rightarrow f$  uniformly on  $S$ .

Proof: WLOG: assume  $f_n \downarrow$  i.e.

$$f_n(x) \geq f_{n+1}(x) \quad \forall x \in S, n \geq 1.$$

For series of  $f_n$ 's,  $\{f_n\}$   
we know if  $f_n = f$  unif.  
&  $f_n$  Cont., then  $f$  is Cont.  
This is a "kind of" converse.

$$\text{Set } F_n = f_n - f \quad \forall n.$$

$$(\because f_n \xrightarrow{p} f)$$

$$\therefore \{F_n\} \subseteq C(S), \quad F_n \downarrow, \quad F_n \geq 0.$$

$$\text{i.e. } F_n(x) \geq F_{n+1}(x) \geq 0 \quad \forall x \in S, n \geq 1.$$

Recall ~~Set~~  $\|F_n\| = \sup \{F_n(x) : x \in S\} = \|F_n\|, \forall n.$

Claim:  ~~$\|F_n\| \rightarrow 0$~~   $\|F_n\| \rightarrow 0.$

Let  $\varepsilon > 0$ . For each  $n \in \mathbb{N}$ , define

$$O_n := F_n^{-1}(-\infty, \varepsilon) = \{x \in S : F_n(x) < \varepsilon\}$$

$\because F_n \in C(S)$ , we have that  $O_n$  open in  $S$   $\forall n$ .

$$\text{Also } F_n \downarrow \Rightarrow O_{n+1} \supseteq O_n \quad \forall n.$$

$\because f_n(x) \rightarrow f(x) \quad \forall x \in S$ , it follows that

$$F_n(x) \rightarrow 0 \quad \forall x \in S.$$

$\therefore$  For each  $x \in S$ ,  $\exists N_x \in \mathbb{N}$  s.t.

$$F_{N_x}(x) < \varepsilon.$$

$$\Rightarrow x \in O_{N_x}.$$

$\therefore \forall x \in S, \exists N \in \mathbb{N}$  s.t.  $x \in O_N$ .

$$\Rightarrow \bigcup_{n=1}^{\infty} O_n = S.$$

$\therefore \{O_n\}$  an open cover of  $S$ .



But  $S$  is Compact.

$$\Rightarrow \exists N \in \mathbb{N} \text{ s.t. } \bigcup_{n=1}^N O_n = S.$$

i.e.  $O_N = S. \quad \leftarrow \because O_n \uparrow.$

i.e.  $\{x \in S : F_N(x) < \varepsilon\} = S.$

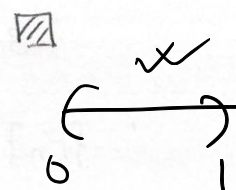
$$\Rightarrow F_N(x) < \varepsilon \quad \forall x \in S.$$

$$\Rightarrow \|F_N\| < \varepsilon.$$

$\because \varepsilon > 0$  is arb.  
it follows...

But  $F_n \downarrow \Rightarrow \|F_n\| \rightarrow 0.$

$$(\because \|F_n\| \leq \|F_N\| < \varepsilon \quad \forall n \geq N).$$



Remark:

①  $S$  is Compact is necessary:

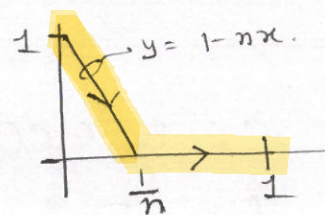
$$f_n(x) = x^n \text{ on } (0, 1).$$

$$\therefore f_n \downarrow \text{ s.t. } f_n \xrightarrow{p} 0. \quad \text{But } f_n \not\rightarrow 0 \text{ unif. on } (0, 1).$$

Continuity of

②  $\forall f$  ( $= \lim f_n$ , pointwise) is continuous is also necessary:

$$f_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq 1/n \\ 0 & 1/n < x \leq 1. \end{cases}$$



$$\frac{x}{1/n} + \frac{y}{1} = 1 \Rightarrow y = 1 - nx$$

That's the idea.

$$\therefore f_n \xrightarrow{p} f$$

where  $f(x) = \begin{cases} 1 & x = 0. \\ 0 & x \in (0, 1]. \end{cases}$

$$\therefore f \notin C[0, 1].$$

$$\text{ s.t. } f_n \not\rightarrow f \text{ unif. as } \|f_n - f\| = 1 \quad \forall n.$$

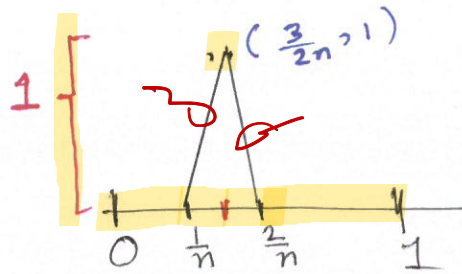
$$\Rightarrow \|f_n - f\| \not\rightarrow 0.$$

□



③ monotonicity of  
 $f_n$  ~~monotonic~~ is also necessary:

Define  $f_n : [0,1] \rightarrow \mathbb{R}$  by



$$f_n(x) = \begin{cases} ?? \\ \end{cases}$$

$\therefore f_n \in C[0,1]$  &  $f_n$  not monotone.

Also  $f_n \rightarrow 0$  pointwise but

$$\|f_n\| = 1 \quad \forall n \Rightarrow f_n \not\rightarrow 0 \text{ unif.}$$

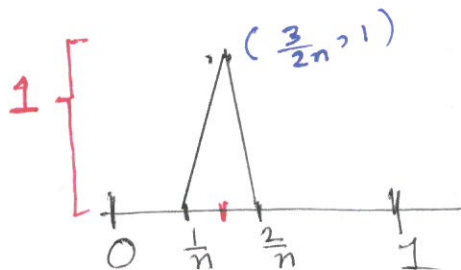
□

— x —.



(3) monotonicity of  
 $f_n$  ~~monotonic~~ is also necessary:

Define  $f_n : [0,1] \rightarrow \mathbb{R}$  by



$\therefore f_n \in C[0,1]$  &  $f_n$  not monotone.

Also  $f_n \rightarrow 0$  pointwise but

$$\|f_n\| = 1 \quad \forall n \Rightarrow f_n \not\rightarrow 0 \text{ unif.}$$

□

— x —

Recall: (Dirichlet's test for convergence)

If  $\sum_{n=1}^{\infty} a_n$  has bounded partial sums (i.e.  $\{S_n\}$  is bdd

where  $S_n = \sum_{j=1}^n a_j$  &  $b_n \downarrow 0$ , then  $\sum a_n b_n$  is convergent.

Also: Abel's test: Let  $\sum a_n$  converges &  $\{b_n\}$  is bdd & monotonic.

Then  $\sum a_n b_n$  converges.

⚡ We will reprove them:  
 But similar technique!!

The  $f_n$ 's theoretic counter parts:

Thm: (Abel's test) Let  $\sum f_n$  converges uniformly on  $S$ , & let  $\{g_n\}$  be uniformly bdd monotone Seqn of ~~real~~  $f_n$ 's on  $S$ . Then  
 $\sum f_n g_n$  converges uniformly on  $S$ .

Proof: Set  $S_n(x) := \sum_{j=1}^n f_j(x) \leftarrow n\text{-th partial sum of } \sum f_n(x).$   
 $\forall x \in S.$

Then  $\forall m > n \geq 1$ , we have:

$$\sum_{j=n+1}^m f_j(x) g_j(x) = (S_m(x) - S_n(x)) g_{n+1}(x) + \sum_{j=n+1}^m (S_m(x) - S_j(x)) (g_{j+1}(x) - g_j(x)).$$

$\leftarrow (S_m - S_n)(x)$

$\forall x \in S.$

$\leftarrow$  Abel's partial summation formula.  
 [HW: Easy to prove.]

Let  $\varepsilon > 0$ .  $\because \sum f_n$  converges unif.  $\exists N \in \mathbb{N}$  s.t.

$$\|S_m - S_n\| < \varepsilon \quad \forall m > n \geq N.$$

$\leftarrow$  Cauchy criterion

Also,  $\{g_n\}$  is uniformly bdd,  $\exists M > 0$  s.t.

$$\|g_n\| < M \quad \forall n \geq 1.$$

$\forall x \in S,$   
 $\& m > n \geq N$

$$\left| \sum_{j=n+1}^m f_j(x) g_j(x) \right| \leq |S_m(x) - S_n(x)| |g_{n+1}(x)| + \sum_{j=n+1}^m |S_m(x) - S_j(x)| |g_{j+1}(x) - g_j(x)|$$

$$< \varepsilon \times M + \varepsilon \sum_{j=n+1}^m |g_{j+1}(x) - g_j(x)|$$

$\leftarrow$  Needs to a fix!!  
 $\leftarrow$  We need to fix this!!

$\therefore \{g_n\}$  is monotonic,  $\sum_{j=n+1}^m |g_{j+1}(x) - g_j(x)|$

is a telescoping sum, &

$$\sum_{j=n+1}^m |g_{j+1}(x) - g_j(x)| = |g_{m+1}(x) - g_{n+1}(x)|$$

$$\leq 2M.$$

$\therefore \forall m > n \geq N$  &  $x \in S$ , we have:

$$\left| \sum_{j=n+1}^m f_j(x) g_j(x) \right| < M \times \varepsilon + \varepsilon \times 2M.$$

$$= (3M) \times \varepsilon.$$

$$\Rightarrow \left\| \sum_{j=n+1}^m f_j g_j \right\| < (3M) \times \varepsilon.$$

$$\forall m > n \geq N.$$

$\therefore$  By Cauchy Criterion:  $\sum f_n g_n$  Converges uniformly.



eg:

Consider  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$ .

Claim: This converges uniformly on  $[0, \infty)$ .

$x=0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  NOT A.C.  
M-test is not useful



(40)

Set  $f_n(x) := \frac{(-1)^n}{n} \quad \forall x \in [0, \infty)$   
 (\*)  $\leftarrow$  constant  $f_n$ 's

$\therefore \sum \frac{(-1)^n}{n}$  is convergent,  $\sum f_n$  is u.c. on  $[0, \infty)$ .

Next, set  $g_n(x) = e^{-nx} \quad \forall x \geq 0, n \geq 1$ .

$\Rightarrow \|g_n\| \leq 1 \quad \forall n$  on  $[0, \infty)$ .

$\& \quad g_n \downarrow$

$\therefore$  By Abel's test:  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$  is u.c.

#  $f_n \equiv \text{const.}$  occurs in "most" practical problems. (\*)

(eg:)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} |x|^n$  is u.c. on  $[-1, 1]$ .

$f_n(x) := \frac{(-1)^n}{n}$  on  $[-1, 1] \quad \forall n$ .

$\& \quad g_n(x) = |x|^n \quad \forall x \in [-1, 1] \quad \& \quad \forall n$ .

$\therefore \sum f_n$  is u.c.  $\& \quad g_n \downarrow, \|g_n\| \leq 1$ .

$\therefore$  Abel's test  $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} |x|^n$  is u.c. on  $[-1, 1]$ .

□

Recall: (Abel's lemma)  
 [Page-85] If  $\alpha \leq \sum_{j=1}^m w_j \leq \beta \quad \forall m=1, \dots, n,$

Then  $\forall$  decreasing  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ , we have:

$$a_1 \alpha \leq \sum_{j=1}^n a_j w_j \leq a_1 \beta.$$

Thm: (Dirichlet test for uniform convergence)

Let  $\{f_n\}, \{g_n\}$  be sequences of f.n's on  $S$ . ~~Let~~ Suppose:

(i) partial sums of  $\sum f_n$  are uniformly bdd on  $S$ .

(ii)  $g_n \downarrow$   ~~$g_n \geq 0$~~   ~~$g_n \downarrow 0$~~

(iii)  $g_n \rightarrow 0$  uniformly on  $S$ .

Then  $\sum f_n g_n$  is uniformly convergent on  $S$ .

Proof: Set  $S_n(x) := \sum_{j=1}^n f_j(x) \quad \forall x \in S, n \in \mathbb{N}.$

(i)  $\Rightarrow \exists M > 0$  s.t.  $\|S_n\| \leq M \quad \forall n \geq 1.$

Now (ii)  $\Rightarrow g_1(x) \geq g_2(x) \geq \dots \geq 0.$



$\forall m > n \geq 1$ , we have

$$\|S_m - S_n\| \leq \|S_m\| + \|S_n\| \leq 2M.$$

$$\Rightarrow \left\| \sum_{j=n+1}^m f_j \right\| \leq 2M.$$

$$\Rightarrow -2M \leq \sum_{j=n+1}^m f_j(x) \leq 2M$$

$$\forall m > n \geq 1 \\ \forall x \in S.$$

Also, (ii)  $\Rightarrow g_1(x) \geq g_2(x) \geq \dots \geq 0.$

$\therefore$  By Abel's lemma:

$$-2M g_{n+1}(x) \leq \sum_{j=n+1}^m f_j(x) g_j(x) \leq 2M g_{n+1}(x).$$

pointwise

$$\Rightarrow \left| \sum_{j=n+1}^m f_j(x) g_j(x) \right| \leq 2M g_{n+1}(x).$$

$$\forall x \in S.$$

i.e.  $\left\| \sum_{j=n+1}^m f_j g_j \right\| \leq 2M \|g_{n+1}\|$

$$\forall m > n \geq 1.$$

Let  $\varepsilon > 0.$

$\therefore g_n \rightarrow 0$  uniformly,  $\exists N \in \mathbb{N}$  s.t.

$$\|g_j\| < \frac{\varepsilon}{2M} \quad \forall j \geq N.$$

$\therefore \forall m > n \geq N$ , we have:

$$\left\| \sum_{j=n+1}^m f_j g_j \right\| \leq \underbrace{2M}_{\text{over}} \|g_{n+1}\|$$

$$< 2M \times \frac{\varepsilon}{2M} = \varepsilon.$$

$\therefore$  By Cauchy criterion:  $\sum_{j=1}^{\infty} f_j g_j$  is uniformly

HW:  $\forall x \in \mathbb{R}$ ,  $\sum_{n \geq 1} 2 \sin \frac{x}{2} \times [\cos x + \cos 2x + \dots + \cos nx]$  Convergent in  $S$ .

$$= \sin \left( n + \frac{1}{2} \right) x - \sin \frac{x}{2}$$

eg: Consider the series:  $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$ .

Important example.

This series converges on  $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$ .

Set  $f_n(x) = \cos nx$ .  $\forall n, x \in \mathbb{R}$ .

~~Indeed~~  $|S_n(x)| = |\cos x + \cos 2x + \dots + \cos nx|$

$$= \left| \frac{\sin \left( \left( n + \frac{1}{2} \right) x \right) - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} \right|$$

$\forall x \neq 2n\pi$

$$\leq \frac{1}{\left| \sin \frac{x}{2} \right|} \quad \forall n \geq 1.$$

i.e.  $|s_n(x)| \leq \frac{1}{|\sin \frac{x}{2}|} \quad \forall n \in \mathbb{N}$   
 $\& x \neq 2n\pi.$

(44)

————— (\*)

$\therefore$  For each fixed  $x \in \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$ ,

$\{s_n(x)\}$  is uniformly bdd.

Also,  $\{\frac{1}{n}\} \downarrow$ ,  $\& \frac{1}{n} \rightarrow 0.$

$\therefore$  By the Dirichlet test (applied to series of real nos.)

$\sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)$  Converges.  $\forall x \in \mathbb{R} \setminus \{2n\pi\}$

-HW-

(1/4)

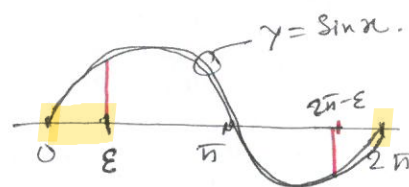
$\sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$

————— 1

————— 11 ———

Something more is true:

Let  $0 < \epsilon < 2\pi.$



See this for  $y = \sin \frac{x}{2}$

Then for  $x \in [\epsilon, 2\pi - \epsilon] \mapsto \sin \frac{x}{2},$

the absolute minimum value is assumed at  $x = \epsilon$  or

$x = 2\pi - \epsilon.$  i.e.

$$|\sin \frac{x}{2}| \geq |\sin \frac{\epsilon}{2}| \Rightarrow \frac{1}{|\sin \frac{x}{2}|} \leq \frac{1}{|\sin \frac{\epsilon}{2}|}$$

$\forall x \in [\epsilon, 2\pi - \epsilon]$

$\therefore (*) \Rightarrow |s_n(x)| \leq \frac{1}{|\sin \frac{\epsilon}{2}|} \quad \forall x \in [\epsilon, 2\pi - \epsilon] \quad \forall n.$

$\epsilon \rightarrow 0$

is  $[\epsilon, 2\pi - \epsilon]$

45

Jaydeb Sarkar.

Then, with  $g_n(x) = \frac{1}{n}$ ,  $x \in [\epsilon, 2\pi - \epsilon]$ , we conclude by the (full) Dirichlet test, that

$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$

converges uniformly on  $[\epsilon, 2\pi - \epsilon]$

$\forall 0 < \epsilon < 2\pi$

□

Then, with  $g_n(x) = \frac{1}{n}$ ,  $x \in [\varepsilon, 2\pi - \varepsilon]$ , we conclude by the (full) Dirichlet test, that

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx \quad \text{converges uniformly on } [\varepsilon, 2\pi - \varepsilon].$$

$\forall 0 < \varepsilon < 2\pi.$

□

(eg: Let  $p > 1$ . Then, by M-test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx$$

converges uniformly to some fn  $f(x)$  on all of  $\mathbb{R}$ .

$\therefore$  the convergence is unif. on  $\mathbb{R}$ , ~~in~~ & since

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \quad (x \in \mathbb{R}),$$

by term-by-term integration:

$$\int_0^{\pi/2} f = \int_0^{\pi/2} \left( \sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \right) dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^p} \int_0^{\pi/2} \cos nx \, dx.$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \sin \left( \frac{n\pi}{2} \right).$$

$$\Rightarrow \int_0^{\pi/2} \left( \sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{p+1}}. \quad \forall p > 1.$$

Next, Consider  $p \in (0, 1]$ .

Then, as in the previous example, use the Dirichlet test to

Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \text{ converges unif. on } [\varepsilon, 2\pi - \varepsilon]$$

$\forall \varepsilon \in (0, 2\pi)$   
 $\nexists p \in (0, 1]$

□

Eg:

Let  $0 < \varepsilon < 1$ . We know:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

uniformly on  $[-\varepsilon, \varepsilon]$

$\therefore \forall x \in [-\varepsilon, \varepsilon]$ , we have:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ on } (-1, 1)$$

$$\int_0^x \left( \sum_{n=0}^{\infty} t^n \right) dt = \int_0^x \frac{1}{1-t} dt$$

$$\Rightarrow \sum_{n=0}^{\infty} \left( \int_0^x t^n dt \right) = -\log(1-x)$$

BTW: What is the defn. of log?

$$\Rightarrow -\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \log(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \forall x \in [-\varepsilon, \varepsilon]$$

Thus:  $\log(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \forall x \in (-1, 1)$

$\nexists$  the convergency is unif. on  $[-\varepsilon, \varepsilon]$   
 $\forall 0 < \varepsilon < 1$



eg:

Note that:

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \quad \forall x \in (-1, 1).$$

uniformly on all compact subsets of  $(-1, 1)$ .

$\therefore \forall x \in (-1, 1)$ , we have:

why?  $\rightarrow$

$$\begin{aligned} \int_0^x \frac{1}{1+t^2} dt &= \int_0^x \left( \sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

$$\Rightarrow \boxed{\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}}$$

$\forall x \in (-1, 1)$   
 $\&$  unif. on  $[-\varepsilon, \varepsilon]$   
 $\forall 0 < \varepsilon < 1$ .

□

# In particular: if  $x = \frac{1}{2}$  &  $x = \frac{1}{3}$ , then

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right) x \frac{(-1)^n}{2n+1}$$

$$\tan^{-1} \left( \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} \right) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\Rightarrow \pi = 4 \times \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left( \frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right)$$

□

eg: (Derivatives).

Consider the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right).$$

$$x=0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Alternating series.  
It converges.

$$\text{Also, } \frac{d}{dx} \left( \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right) \right) = \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{x}{n}\right).$$

Now by M-test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{x}{n}\right)$$

Converges unif. on  $\mathbb{R}$ .

$$\therefore f(x) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right)$$

defines a diff.  $f_n$  on  $\mathbb{R}$ . & the above series

converges unif. on  $\mathbb{R}$ . Moreover:

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{x}{n}\right) \quad \forall x \in \mathbb{R}.$$

Also, u.c.

Sequences of improper integrals:

Let  $\{f_n\} \subseteq \mathcal{R}[a, \infty)$

& let  $f_n \rightarrow f$  or  $\sum f_n = f$

pointwise or uniformly.

$$\text{Q: } \lim_{n \rightarrow \infty} \int_a^{\infty} f_n = \int_a^{\infty} f \quad \text{or} \quad \sum \int_a^{\infty} f_n = \int_a^{\infty} f \quad ??$$

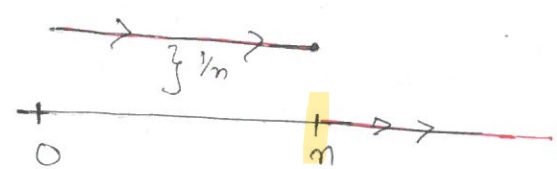
# Of course, we need the assurance of  $f \in \mathcal{R}[a, \infty)$  !!

Note: perhaps with pointwise convergence, there is no/less hope.  
What about u.c. ??

↓ Again, no hope.

eg:

$\forall n \in \mathbb{N}$ , define  $f_n : [0, \infty) \rightarrow \mathbb{R}$  by:



$$\text{i.e. } f_n(x) = \begin{cases} \frac{1}{n} & 0 \leq x \leq n \\ 0 & x > n. \end{cases}$$

Clearly,  $f_n \in \mathcal{R}[0, \infty)$  &  $\int_0^\infty f_n = 1 \quad \forall n$ .

Also,  $\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x \in [0, \infty)$  & uniformly.

$\therefore f_n \rightarrow 0$  unif. on  $[0, \infty)$ .

$$\therefore \lim_{n \rightarrow \infty} \int_0^\infty f_n = 1 \neq 0 = \int_0^\infty \lim_{n \rightarrow \infty} f_n. \quad \left[ \because \|f_n\| \leq \frac{1}{n} \right]$$

# Observe that  $\nexists g \in \mathcal{R}[0, \infty)$  s.t.

$$f_n(x) \leq g(x) \quad \forall x \in [0, \infty) \quad \forall n \in \mathbb{N}.$$

And this lack of dominance is the key!

For instance (a baby version of dominated convergence thm). (DCT)

Thm (DCT).  $\leftarrow$  Very useful theorem!!

Let  $\{f_n\} \subseteq C[a, \infty)$  s.t.  $f_n \rightarrow f$  unif. on  $[a, b]$   
 $\forall b > a$ .

Let  $g \in C[a, \infty)$   ~~$\cap R[a, \infty)$~~  s.t.  $\int_a^\infty g$  exists  $\neq$   
 $|f_n(x)| \leq g(x) \quad \forall x \in [a, \infty), n \geq 1$ .

Then 
$$\lim_{n \rightarrow \infty} \int_a^\infty f_n = \int_a^\infty f$$

$f(x) = x$   
 $\int_a^\infty x$  DNE

Proof: HW.

## § Power Series.

Def: Let  $c \in \mathbb{R}$  &  $\{a_n\}_{n \geq 0} \subseteq \mathbb{R}$ . The formal sum

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{--- (P)}$$

is called a power series about c (or center c) <sup>with</sup>

with coefficients  $\{a_n\}$ .

# eg: Polynomials are power series & admit (P) for any  $c \in \mathbb{R}$ ! Q: Given a polynomial, how you write it as (P)?



## Note 8

(51)  
Jaydeb Sarkar

- (1) Setting  $f_n(x) = a_n(x-c)^n$ , we realize that the P.S. (P) is perhaps <sup>the</sup> simplest series of  $f_n$ 's  $\sum_{n=0}^{\infty} f_n(x)$ .
- (2) Our interest: Convergence (pointwise/absolute/uniform) of the series (P) & then determine the limit  $f_n$ .
- (3) Adopt all <sup>the</sup> ideas of series of  $f_n$ 's.
- (4) If  $S \subseteq \mathbb{R}$  s.t. (P) converges  $\forall x \in S$ , then  $\exists f: S \rightarrow \mathbb{R}$  s.t.  $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \quad \forall x \in S$ .

# Clearly,  $S \neq \emptyset$  as  $c \in S$ !

- (5) eg:  $\sum_{n=0}^{\infty} x^n$  is a P.S. <sup>with</sup> center at 0 & coefficients  $\{1\}$ .

We know:  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$  ;  $x \in [0, 1)$   
 $\quad \quad \quad \downarrow$   $\quad \quad \quad \downarrow$   
 $\quad \quad \quad f(x)$   $\quad \quad \quad S$   
 $\quad \quad \quad \downarrow$   
 $\quad \quad \quad$  Defined only on  $(-1, 1)$   $\quad \quad \quad$  Defined on  $\mathbb{R} \setminus \{1\}$

Geometric series.

$\therefore \{P.S.\} \longleftrightarrow \{\text{Functions}\}$

We need to understand this link (if any).  
& Corresponding domain of definitions!!

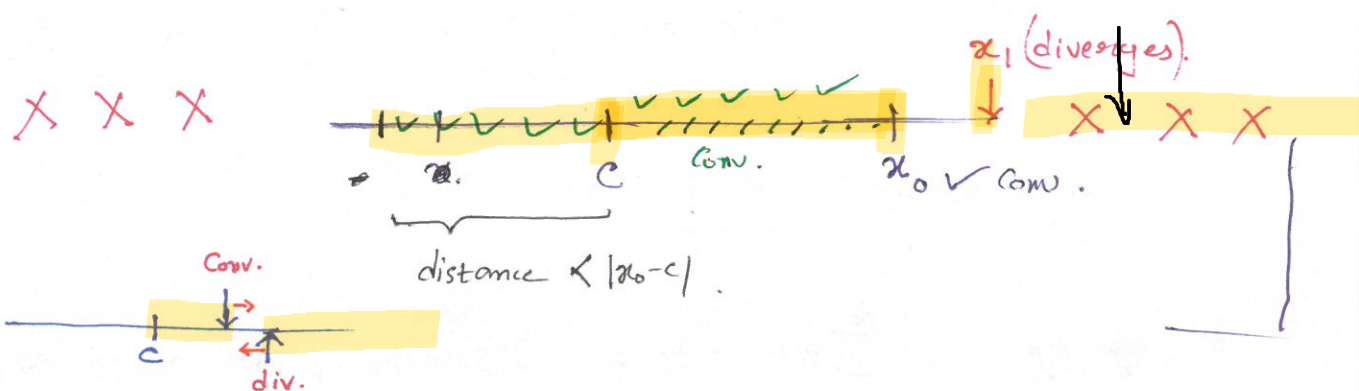
The following is remarkable:

Thm: Consider the P.S.  $\sum_{n=0}^{\infty} a_n(x-c)^n$  — (P).

(1) If (P) converges for some  $x_0 \in \mathbb{R}, \{x_0\}$ , then it converges absolutely for all  $x$  s.t.  $|x-c| < |x_0-c|$ .

(2) If (P) diverges at  $x_1 \in \mathbb{R}$ , then it diverges  $\forall x \in \mathbb{R}$  s.t.  $|x-c| > |x_1-c|$ .

$\therefore$  We have the following picture:



Proof: Let (P) converges at  $x = x_0$ . (Also,  $x_0 \neq c$ ).

$$\Rightarrow \sum_{n=0}^{\infty} a_n(x_0 - c)^n \text{ Converges.}$$

← A series of real no's.

$$\Rightarrow a_n(x_0 - c)^n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$\therefore$  For  $\varepsilon = \frac{1}{2}$ ,  $\exists N \in \mathbb{N}$  s.t.

$$|a_n(x_0 - c)^n| < \frac{1}{2} \quad \forall n \geq N.$$

Let  $x \in \mathbb{R}$  & suppose  $|x-c| < |x_0-c|$ .



$$\therefore |a_n(x-c)^n| = |a_n(x_0-c)^n| \times \left| \frac{x-c}{x_0-c} \right|^n$$

$$\forall n \geq N \quad \frac{1}{2} \times \left| \frac{x-c}{x_0-c} \right|^n$$

But  $|x-c| < |x_0-c| \Rightarrow \left| \frac{x-c}{x_0-c} \right| =: r$  for some  $r \in (0,1)$ .

$$\therefore |a_n(x-c)^n| < \frac{1}{2} r^n \quad \forall n \geq N$$

$\therefore$  By Comparison test,  $\sum_{n=0}^{\infty} |a_n(x-c)^n|$  converges.

i.e. (P) is A.C.  $\forall x \in \mathbb{R}$  s.t.  $|x-c| < |x-x_0|$ . ✓

For (2): Let  $x_1 \in \mathbb{R}$  s.t.  $|x_1-c| > |x_0-c|$ .

If (P) converges at  $x$ , then by (1), (P) will

converge at  $x_1$   $\rightarrow \leftarrow$ .

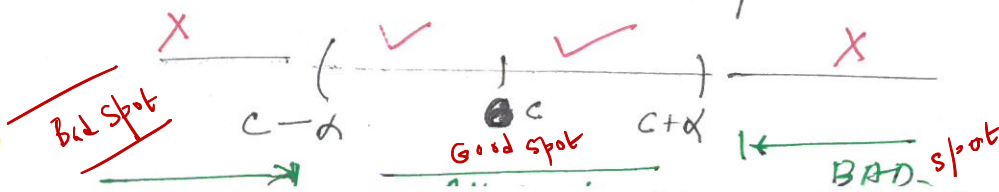
□

# The above result is curious: We need to find the

"maximum" of  $x_0$  s.t.  $\sum a_n(x_0-c)^n$  converges &

"min. of  $x_1$  s.t.  $\sum a_n(x_1-c)^n$  diverges. ✓

If there is such a min-max, say  $\alpha$ , then



Q: How to find/determine that "min-max"?

Recall: (Root test)

Let  $\sum_{n=0}^{\infty} a_n$  be a series of trv no's.

Set  $\rho = \limsup \sqrt[n]{|a_n|}$ .

possibly  $\rho = +\infty$ .

- (1) If  $\rho < 1$ , then the series converges.
- (2) If  $\rho > 1$ , then the series diverges.
- (3) If  $\rho = 1$ , the test is inconclusive.

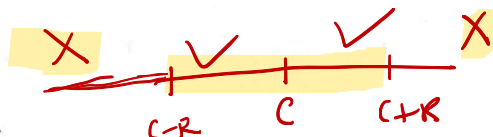
CONVENTION:  $\frac{1}{\infty} = 0$ ;  $\frac{1}{0} = \infty$ .

HW

Thm: (Cauchy - Hadamard Thm).

Consider the P.S.  $\sum_{n=0}^{\infty} a_n(x-c)^n$ . & Set

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$



Then the P.S. converges  $\forall x \in \mathbb{R}$  s.t.  $|x-c| < R$ .

& diverges  $\forall x \in \mathbb{R}$  s.t.  $|x-c| > R$ .

Proof:

Observe that  $\forall x \in \mathbb{R}$ ,

$$\limsup |a_n(x-c)^n|^{1/n} = \frac{|x-c|}{R}.$$

$\therefore$  The result follows from the Root test.

□

Def. Given a P.S.  $\sum_{n=0}^{\infty} a_n(x-c)^n$ , the number  $R \in \mathbb{R} \cup \{\infty\}$  is called the radius of convergence, where

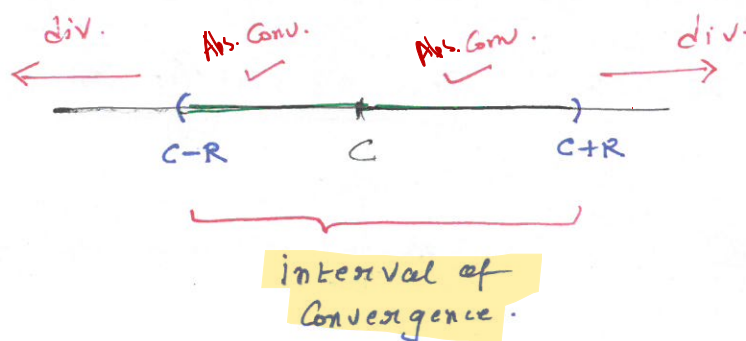
$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}.$$

Recall:  $\frac{1}{\infty} = 0$  &  $\frac{1}{0} = \infty$

Moreover, ~~Reals~~  $(c-R, c+R)$  is called the interval of convergence of the P.S.

#  $\therefore$  By Cauchy-Hadamard thm., for  $\sum_{n=0}^{\infty} a_n(x-c)^n$  & with

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|},$$



\* No conclusion about end points  $\{c \pm R\}$ .

# If  $R=0$ , then the series converges only at  $x=c$ .

Remark: Let  $\{\alpha_n\}$  be a seqn. of pos. no's. Then:

$$\liminf \frac{\alpha_{n+1}}{\alpha_n} \leq \liminf \sqrt[n]{\alpha_n} \leq \limsup \sqrt[n]{\alpha_n} \leq \limsup \frac{\alpha_{n+1}}{\alpha_n}.$$

$\therefore$  If  $\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n}$  exists, then  $\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\alpha_n}.$

Cor: If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then for  $\sum_{n=0}^{\infty} a_n (x-c)^n$ , the radius of convergence is given by:

useful  
for computation.

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

eg: ① Let  $p \in \mathbb{R}[x]$ . Fix  $c \in \mathbb{R}$ .

$$\Rightarrow p = \sum_{n=0}^N a_n (x-c)^n. \quad (*)$$

$$\therefore \frac{1}{R} = \limsup \sqrt[n]{|a_n|} = 0$$

$$\Rightarrow R = \infty.$$

$\therefore$  the r.o.c. (radius of convergence) of  $(*)$  is  $\infty$ .

$\Rightarrow \mathbb{R}$  is the interval of convergence.

$\&$  the p.s. conv. absolutely on  $\mathbb{R}$ .

②

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$c=0, a_n = \frac{1}{n!} \quad \therefore \text{A p.s.}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

$$\Rightarrow R = \infty.$$

$\therefore$  r.o.c. i.e.  $R = \infty$   $\&$   $\mathbb{R}$  is the interval of conv.

We define:  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R}.$

③  $\frac{1}{3} - x + \frac{x^2}{3^2} - \frac{x^3}{3^3} + \frac{x^4}{3^4} - x^5 + \dots$

$$a_n = \begin{cases} \frac{1}{3} & n=0 \\ \frac{1}{3^n} & n \text{ even.} \\ -1 & n \text{ odd.} \end{cases}$$

$$\Rightarrow \limsup \sqrt[n]{|a_n|} = 1$$

$$\Downarrow \\ R = 1$$

~~$\therefore \sum_{n=0}^{\infty} \frac{1}{3^n} x^n$~~

$\therefore$  Radius of Conv. is 1.

④  $\sum_{n=0}^{\infty} \frac{n!}{(n+1)^{n+1}} x^n$

← A p.s. about 0.  
with coefficients  $\left\{ \frac{n!}{(n+1)^{n+1}} \right\}$ .

Here  $C=0$ ,  $a_n = \frac{n!}{(n+1)^{n+1}}$

$$\therefore \left| \frac{a_n}{a_{n+1}} \right| = \frac{n!}{(n+1)!} \times \frac{(n+2)^{n+2}}{(n+1)^{n+1}}$$

$$= \frac{1}{n+1} \times \frac{(n+2)^{n+2}}{(n+1)^{n+1}} = \left( \frac{n+2}{n+1} \right)^{n+2}$$

$$= \left( 1 + \frac{1}{n+1} \right)^{n+2} \longrightarrow e \text{ as } n \rightarrow \infty.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{e}$$

↑  
why??

$\Rightarrow$  Radius of Convergence is  $e$ .

$\therefore$  The P.S. Converges absolutely in  $\{x \in \mathbb{R} : |x| < e\}$ .



(5)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$  .  $\left( = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right)$

$$\therefore \frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

$$\Rightarrow R = 1$$

$\therefore (-1, 1)$  is the interval of convergence.

However (END POINTS): If  $x=1$ , we have  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$   
which is convergent.

If  $x=-1$ , then we have

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \leftarrow \text{divergent}$$

# [ ~~end points~~ end points are ~~also~~ not determined: rather it depends case by case. ]

— x —

Remark: WLOG: we may simply study P.S. with center 0.

$\therefore$  From now on:  $\sum_{n=0}^{\infty} a_n x^n$

# Recall: If  $\sum_{n=0}^{\infty} a_n x^n$  Converges at  $x_0 \in \mathbb{R}$  ( $x_0 \neq 0$ ),  
then  $\sum_{n=0}^{\infty} a_n x^n$  Converges  $\forall x \in \mathbb{R}$  s.t.  
 $|x| < |x_0|$ .



The idea was:  $\sum_{n=0}^{\infty} a_n x^n$  converges at  $x_0$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x_0^n \text{ Conv.}$$

$$\Rightarrow \underline{a_n x_0^n \rightarrow 0} \Rightarrow \text{for } \varepsilon = \frac{1}{2}, \exists N \in \mathbb{N} \text{ s.t.}$$

$$\underline{|a_n x_0^n| < \frac{1}{2} \quad \forall n \geq N.}$$

Now if  $|x| < |x_0|$ , then

$$|a_n x^n| < |a_n x_0^n| \quad \forall n \geq 1$$

$$\Rightarrow \sum_{n=0}^{\infty} |a_n x^n| < \sum_{n=0}^{\infty} |a_n x_0^n|$$

$$\therefore \underline{\sum |a_n| |x_0^n| < \infty}, \quad \text{free } x < \infty.$$

By M-test,  $\sum_{n=0}^{\infty} a_n x^n$  is uniformly convergent on  $[-x_0, x_0]$ .

Thus, we have the following:

# Thm: Let  $R$  = radius of convergence of the P.S.  $\sum_{n=0}^{\infty} a_n x^n$ .  
Then  $\sum_{n=0}^{\infty} a_n x^n$  is u.c. on all closed intervals  $\subseteq (-R, R)$ .

# on  $(-R, R)$   
P.S. is A.C. & unif. on all comp. subsets of  $(-R, R)$ .

↑  
Very useful property.

↑  
[Compact Subsets]

The following is now easy:

Thm: Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  on  $(-R, R)$ .

the radius of convergence.  
Assume  $R \neq 0$ .

Then  $\forall x \in (-R, R)$ ,

$$\int_0^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

on Compact Subsets.

Proof:  $\therefore$  term-by-term int. is allowed for u.c. Series.

Q: What about derivatives of P.S.?

$R$  = radius of convergence, always.

Notation:

Remark: Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  on  $(-R, R)$ .

$$\left( \therefore \frac{1}{R} = \limsup \sqrt[n]{|a_n|} \right)$$

Def: Given a P.S.  $\sum_{n=0}^{\infty} a_n x^n$ , the derived series is the

new P.S.

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

← The term-by-term derivatives.

Thm: Let  $R_d$  = radius of convergence of the derived P.S.  $\sum_{n=1}^{\infty} n a_n x^{n-1}$ .

Then  $R = R_d$ .

$$\left[ R = \frac{1}{\limsup \sqrt[n]{|a_n|}} \right]$$

Proof:

By definition:

$$\frac{1}{R_d} = \limsup \sqrt[n]{n |a_n|}$$

"

$$\sqrt[n]{n} \times \sqrt[n]{|a_n|}$$

∴ we know  $\sqrt[n]{n} \rightarrow 1$  as  $n \rightarrow \infty$ .

$$\therefore \limsup \sqrt[n]{n |a_n|}$$

$$= \lim_{n \rightarrow \infty} \sqrt[n]{n} \times \limsup \sqrt[n]{|a_n|}$$

$$= 1 \times \frac{1}{R}$$

$$\Rightarrow R = R_d$$

If  $\frac{1}{R} < \infty$ .

If  $R = \infty$ :



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Jaydeb Sarkar

Cor: If  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  on  $(-R, R)$ , then  $f$  is diff. on  $(-R, R)$

$$\& \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \forall x \in (-R, R).$$

The derived P.S.

$\therefore \frac{d}{dx}(\text{P.S.}) = \text{derived P.S.}$   
on the same region of conv. interval

Proof:  $\therefore$  The derived P.S. is u.c. on ~~every~~ all closed intervals contained in  $(-R, R)$ , it follows that ~~f(x)~~  $f'(x)$  exists  $\forall x \in (-R, R)$  &

$f'(x)$  = the derived sum.

using derivatives of series of u.c. fns.

Fix  $x \in (-R, R)$ .  
 $\Rightarrow f(x) = \sum f_n(x)$   
diff. on  $[x-s, x+s] \subset (-R, R)$   
for some  $s > 0$ .

Note that  $\sum f'_n(x)$  is u.c. on  $[x-s, x+s] = S$ .  
use Conv. of  $\sum f'_n$ .

Cor: Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  on  $(-R, R)$ . Then  $f$  has derivatives of all orders on  $(-R, R)$ .

Moreover:

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall n \geq 0. \quad \#$$

Proof:  $f^{(n)}(x)$  exists  $\forall n \geq 0$  &  $x \in (-R, R)$  follows from the previous Corollary.

The equality follows from induction.

Remark: The above Corollary  $\Rightarrow$  P.S. is !.

i.e. if  $f: (-R, R) \rightarrow \mathbb{R}$  is a fn. s.t.  $f(x) = \sum_{n=0}^{\infty} a_n x^n$ ,

& if  $f(x) = \sum_{n=0}^{\infty} b_n x^n$  on  $(-R, R)$ , then

$$a_n = b_n \quad \forall n.$$

$\sum a_n x^n$   
 $\sum b_n (x-c)^n$   
#  
 $a_n = b_n$ ?

□

Remark: Suppose  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  on  $(-R, R)$ . (Assume  $R > 0$ ).

Then we already proved:

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall n \geq 0.$$

$$\therefore f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$



on  $(-R, R)$ .

The Taylor series of the fn.  $f$  about 0.

Treat this as the defn. of Taylor series.

Q: Let  $f$  be a fn. that is infinitely diff. ~~at~~ in a nbd of 0, say on  $(-\varepsilon, \varepsilon)$  for some  $\varepsilon > 0$ .

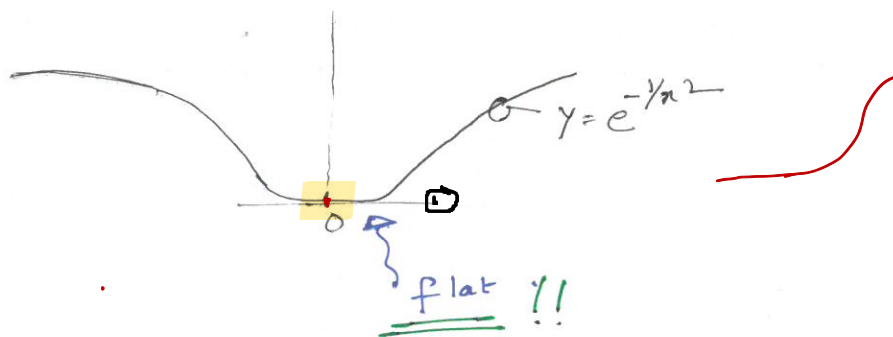
$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \forall x \in (-\varepsilon, \varepsilon)?$$

The Taylor series of  $f$  around 0.  
or  $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Ans: No!!

eg:  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$\sum a_n (x-c)^n$   
 $c \neq 0$   
 $(-R, R)$   
 $a_n = \frac{f^{(n)}(c)}{n!}$





Then  $f$  is infinitely diff. ~~at 0~~, on  $\mathbb{R}$ .

Easy to see  $\forall x \in \mathbb{R} \setminus \{0\}$ .

At  $x=0$ : Check (HW).

Moreover:  $f^{(n)}(0) = 0 \quad \forall n \geq 0$ .

$\Rightarrow$  The Taylor expansion of  $f$  around 0 is:

$$\sum \frac{f^{(n)}(0)}{n!} x^n \equiv 0.$$

$$\Rightarrow f(x) \neq \underbrace{\sum \frac{f^{(n)}(0)}{n!} x^n}_{\equiv 0} \quad \forall x \in (-\varepsilon, \varepsilon) \text{ for any } \varepsilon > 0!!$$

# You will face this in Complex analysis!!

Def: Let  $f: (a, b) \rightarrow \mathbb{R}$  be a fn. We say that  $f$  is analytic at  $c \in (a, b)$  if there is a p.s. about  $c$  that represents  $f$  in a nbd of  $c$ . i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \forall x \in (c-s, c+s) \text{ for some } s > 0.$$

$$\# f \text{ is analytic at } c \iff f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n, \quad \forall x \in (c-s, c+s).$$

(The Taylor series of  $f$  about  $c$ .)

Remark: Clearly, if  $f$  is analytic at  $c$ , then  $f$  is smooth at  $c$  [i.e.  $f^{(n)}(c)$  exists  $\forall n \geq 0$ ].

Of course, smooth  $\not\Rightarrow$  analytic.

"Complex analysis"

eg:  $f(x) = \frac{1}{1-x}$  is analytic at  $c \in \mathbb{R} \setminus \{1\}$ .

In fact: remember:  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$   $|x| < 1$ .

In general: if  $c \neq 1$ , then:

$$f(x) = \frac{1}{1-c} \left[ \frac{1}{1 - \frac{x-c}{1-c}} \right]$$

$$= \frac{1}{1-c} \times \sum_{n=0}^{\infty} \left( \frac{x-c}{1-c} \right)^n \quad \forall \left| \frac{x-c}{1-c} \right| < 1.$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-c)^{n+1}} (x-c)^n \quad \forall x \in \mathbb{R} \text{ s.t. } |x-c| < |1-c|.$$

$\therefore f$  is defined on all of  $\mathbb{R} \setminus \{1\}$ , but, the above equality holds only on  $|x-c| < |1-c|$ .

$\therefore$  The radius of convergence of the above P.S. is  $|1-c|$  & interval of conv. is

$$(c - |1-c|, c + |1-c|).$$

All depends on  $c$ .

[4]



Def: Let  $O \subseteq \mathbb{R}$  be open, & let  $f: O \rightarrow \mathbb{R}$ . We say that  $f$  is analytic on  $O$  if- it is analytic at each  $c \in O$ . i.e.  $\forall c \in O, \exists \delta (= \delta(c)) > 0$  s.t.  $\{a_n\} \subseteq \mathbb{R}$  (again:  $a_n = a_n(c)$ ) s.t.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \forall x \in O \cap (c-\delta, c+\delta).$$

[ $\Leftrightarrow f$  admits p.s./Taylor exp. about  $c \quad \forall c \in O$ ].

Note: Often we say "Real analytic" instead of analytic. But ~~this~~ can wait till Complex analysis.  
real analytic

Eg: Let  $\sum_{n=0}^{\infty} a_n x^n$  be a p.s. with radius of convergence  $R > 0$ .

Then  $f(x) := \sum_{n=0}^{\infty} a_n x^n$  is analytic on  $(-R, R)$ .

WHY?  
Ans.

Thm: Let  $\sum_{n=0}^{\infty} a_n x^n := f(x)$  has radius of convergence  $R > 0$ .

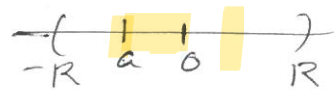
Suppose  $a \in (-R, R)$ . Then the Taylor series expansion of  $f$  about  $a$  is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n,$$

for all  $x \in \mathbb{R}$  s.t.  $|x-a| < R - |a|$ .

Proof: Fix  $a \in (-R, R)$ .

Set  $S = R - |a|$ .



We use  $x^n = (x-a+a)^n = \sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$ .

$\forall n \geq 0$ .

*Handwritten notes in red:*  
 $\sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$   
 $\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$$

Set  $\alpha_{m,n} = \begin{cases} \binom{n}{m} & \text{if } m=0,1,\dots,n \\ 0 & \text{oth. (i.e. } m > n) \end{cases}$

*Handwritten notes in red:*  
 $\sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$   
 $\Rightarrow \sum_{n=0}^{\infty} a_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$   
WAIT

"double series."

*Handwritten notes in red:*  
 $\sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$   
 $\Rightarrow \sum_{n=0}^{\infty} a_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x-a)^m$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \alpha_{m,n} a^{n-m} (x-a)^m$$

Note that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_n \alpha_{m,n} a^{n-m} (x-a)^m|$$
$$= \sum_{n=0}^{\infty} |a_n| \sum_{m=0}^n \binom{n}{m} |a|^{n-m} |x-a|^m$$
$$= \sum_{n=0}^{\infty} |a_n| (|x-a| + |a|)^n$$

Testing A.C. of double series.

If  $|x-a| < S = R - |a| \Rightarrow |x-a| + |a| < R$ .

$\Rightarrow (*)$  Converges.

$\Rightarrow$  The double series in  $(+)$  is absolutely convergent.

*Handwritten notes in red:*  
 $\Rightarrow$  The double series in  $(+)$  is absolutely convergent.

[Convergence of double series will be discussed later.]

[A double series  $\sum a_{m,n}$  is A.C. if  $\sum |a_{m,n}|$  converges]

"Due"

$$\Leftrightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty \Leftrightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{m,n}| < \infty.$$

(57)

AND: In this case: ← "due"/"pending fact":

$$\sum_{m,n} a_{m,n} = \sum_m \sum_n a_{m,n} = \sum_n \sum_m a_{m,n}$$

Change of order of summations

∴ By Changing the order of summation in (†):

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_n \alpha_{m,n} a^{n-m} (x-a)^m.$$

↗ x s.t.  $|x-a| < R-|a|$

$$= \sum_{m=0}^{\infty} \left( \sum_{n=m}^{\infty} a_n \binom{n}{m} a^{n-m} \right) (x-a)^m.$$

$$= \sum_{m=0}^{\infty} \tilde{a}_m (x-a)^m.$$

Recall:  
 $\alpha_{m,n} = \begin{cases} \binom{n}{m} & m \leq n \\ 0 & m > n. \end{cases}$

where  $\tilde{a}_m := \sum_{n=m}^{\infty} a_n \binom{n}{m} a^{n-m} \quad \forall m \geq 0.$

i.e.  $\sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} \tilde{a}_m (x-a)^m$

↖ x s.t.  $|x-a| < \delta = R-|a|.$

⇒  $\sum_{n=0}^{\infty} a_n x^n$  is analytic at a.

□

# Given a fn.  $f: S \rightarrow \mathbb{R}$ , denote by  $Z(f)$  the zero set of  $f$ . i.e.

$$\therefore Z(f) = \{x \in S : f(x) = 0\}.$$

# If  $p \in \mathbb{R}[x]$ , then  $\# Z(p) < \infty$ . if  $p \neq 0$ .

Q: What about  $Z\left(\sum_{n=0}^{\infty} a_n x^n\right)$ ? OR,  $Z(\text{Analytic fn.})$

Ans: "Like" polynomials.

??

→ Next page →

~~Thm: Let  $f: \mathcal{O} \rightarrow \mathbb{R}$  be an analytic fn., where  $\mathcal{O}$  is an open interval.~~

~~Thm: Let  $f: (a, b) \rightarrow \mathbb{R}$  be an analytic fn. If  $Z(f)$  has a limit-point in  $(a, b)$  then  $f \equiv 0$ .~~

~~Proof: Let  $c$  be a limit point of  $Z(f)$  &  $c \in (a, b)$ .~~

~~$\Rightarrow f(c) = 0$ .~~

~~$[\because f \text{ is cont. at } c]$~~

~~$\Rightarrow Z(f)' \subseteq Z(f)$ .~~

~~If possible, let  $f \neq 0$ .~~

~~We know:  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$  on  $(c-s, c+s)$ .~~

~~$\because f \neq 0, \exists$~~

PTO

iso  $Z(f) = \{0\}$



First, observe that if  $f: (a,b) \rightarrow \mathbb{R}$  is analytic, then  $f$  is Cont. on  $(a,b)$  [ $\because f$  is diff. on  $(a,b)$ ]

$$\Rightarrow Z(f) = \underline{f^{-1}(\{0\})} \text{ is a closed set.}$$

$$\therefore Z(f)' \subseteq Z(f)$$

Set of limit points of  $Z(f)$ .

$(a,b)$  is important.  
 $\therefore (0,1) \cup (3,4)$   
 may not work!!

Thm: Let  $f \not\equiv 0$  be an analytic fn on  $(a,b)$ . Then  $Z(f)$  ~~does not~~ cannot have a limit point in  $(a,b)$ .

( $\Leftrightarrow Z(f)$  is a set of <sup>some</sup> isolated points).

Proof:

We prove zeros of  $f$  are isolated.

$$\text{Set } \mathcal{O} := \{x \in (a,b) : f^{(n)}(x) = 0 \quad \forall n = 0, 1, \dots\}$$

$$\therefore \text{If } c \in \mathcal{O}, \text{ then } f^{(n)}(c) = 0 \quad \forall n \geq 0.$$

$$\because f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n \quad \text{in a nbd of } c,$$

it follows that  $f \equiv 0$  in a nbd of  $c$ .

$$\Rightarrow \underline{\mathcal{O} \text{ is open.}} \quad (\text{possibly } \emptyset).$$

Next, assume that  $c \in (a,b) \setminus \mathcal{O}$ .

$$\Rightarrow \exists m \geq 0 \text{ s.t. } f^{(m)}(c) \neq 0.$$

$\{m \text{ may depend on } c\}$

(70)

But  $f^{(m)}$  is also analytic at  $c$ .  $\leftarrow$  Why?

$\therefore f^{(m)}(c) \neq 0$ , by continuity of  $f^{(m)}$  at  $c$ , it follows that  $f^{(m)}(x) \neq 0$  in a nbd of  ~~$c$~~   $c$  contained in  $(a, b) \setminus \mathcal{O}$ .

Recall:  $x_0 \in \mathcal{O}$   
 $\Leftrightarrow f^{(n)}(x_0) = 0$   
 $\forall n \geq 0$

$\Rightarrow (a, b) \setminus \mathcal{O}$  is also open.

\*  $\therefore$  Both  $\mathcal{O}$  and  $(a, b) \setminus \mathcal{O}$  are open.

But  $(a, b)$  is a connected set (or an interval).

$\Rightarrow$  either  $\mathcal{O} = \varnothing$  or  $(a, b) \setminus \mathcal{O} = \varnothing$ .

$\Rightarrow$  either  $\mathcal{O} = \varnothing$  or  $\mathcal{O} = (a, b)$ .  $\Leftrightarrow f \equiv 0$ .

$\therefore$  if  $f \neq 0$ , then zeros of  $f$  are isolated points.

indeed

If  $c \in \mathcal{Z}(f)$ , then by  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$ ,

we know that  $\exists m \in \mathbb{N}$  s.t.  $f^{(m)}(c) \neq 0$

$$0 = f(c) = f^{(1)}(c) = \dots = f^{(m-1)}(c)$$

$$f^{(m)}(c) \neq 0$$

$$\Rightarrow f(x) = (x-c)^m \times \left( \frac{f^{(m)}(c)}{m!} + \frac{f^{(m+1)}(c)}{(m+1)!} (x-c) + \dots \right)$$

$= g$  : A.P.S.  
 with same radius  
 of convergence.

$$= (x-c)^m \times g(x)$$

defined in a nbd of  $c$ .



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Jaydeb Sarkar

But  $f$  is also analytic &  $f(c) \neq 0$ .

( $\because f^{(m)}(c) \neq 0$ )

$\therefore$  By Cont.  $f(x) \neq 0 \forall x$  in a nbd of  $c$ .

$\Rightarrow f(x) \neq 0 \forall x \in$  deleted nbd of  $c$ .

$\Rightarrow c$  is an isolated point.

□

"Identity thm".

Cor: Let  $f, g : (a, b) \rightarrow \mathbb{R}$  analytic. If  $f(z) = g(z)$   
 $\forall z \in A$  s.t.  $A' \cap (a, b) \neq \emptyset$ , then  $f = g$  on  $(a, b)$ .

— x —

Thm: (Abel's thm (1826))

Let  $\sum_{n=1}^{\infty} a_n$  Converges. Then the series  $\sum_{n=0}^{\infty} a_n x^n$  Converges uniformly

Continuous on  $[0, 1]$ .

$f$  Cont. on  $(0, 1]$ .

$f = \sum_{n=0}^{\infty} a_n x^n$

on  $(-R, R)$

$\sum a_n R^n$  Conv. the  $\sum a_n$  is w/ cont.

$(a \in [0, 1])$

Works for  $R > 0$ .

$\therefore \sum a_n$  Conv. it follows that  $\sum_{n=0}^{\infty} a_n x^n$  Conv. on  $(-1, 1]$  & A.C. on  $(-1, 1)$ .

Enough to prove that:

$$\lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n$$

As: If  $f(x) := \sum_{n=0}^{\infty} a_n x^n$ , then  $f$  is a p.s. with radius of conv  $\geq 1$

The above  $\Rightarrow \lim_{n \rightarrow \infty} f(n) = f(1) \Rightarrow f$  is Cont. on  $[0, 1]$ .

$\sum a_n x^n$  is Conv. on  $(-1, 1)$ .  
 $f$  w/ cont. on all closed subsets of  $(-1, 1)$ .

$\sum (-1)^n a_n$  Conv.  $\Rightarrow [-1, 0]$  is u.c.

Proof: Set  $f(x) := \sum_{n=0}^{\infty} a_n x^n$  .  $|x| < 1$  .

Claim:  $\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} a_n$  . We know rad. of conv.  $\geq 1$  .

Set  $\alpha_n := \sum_{k=0}^{n-1} a_k$  . We know  $\alpha_n \rightarrow \sum a_n := \alpha$  .

Also set  $s_n(x) := \sum_{k=0}^{n-1} a_k x^k$  .  $\leftarrow$   $n$ -th partial sum of  $\sum a_n x^n$  .

By Abel's Lemma:  $s_n(x) = \sum_{k=0}^{n-1} \alpha_k (x^k - x^{k+1}) + \alpha_n x^n$  .

$$\Rightarrow s_n(x) = \sum_{k=0}^{n-1} \alpha_k (1-x) x^k + \alpha_n x^n$$

$\downarrow_{n \rightarrow \infty}$

$\therefore \forall x \in (0, 1)$ , as  $n \rightarrow \infty$ , we have:

$$f(x) = (1-x) x \sum_{n=0}^{\infty} \alpha_n x^n$$

$\{a_n\}$  bdd  
 $s_n x^n \rightarrow 0$   
 $a_n x^n \rightarrow 0$

$\left[ \because \alpha_n x^n \rightarrow 0 \right]$   
 as  $\alpha_n \rightarrow \sum a_n$   
 $s_n x^n \rightarrow 0$  .

$$\Rightarrow f(x) - \underbrace{\sum_{n=0}^{\infty} a_n}_{= \alpha} = (1-x) x \sum_{n=0}^{\infty} (\alpha_n - \alpha) x^n$$

$\underbrace{(1-x) x \sum_{n=0}^{\infty} x^n}_{= 1} \times \underbrace{\sum_{n=0}^{\infty} a_n}_{= \alpha}$

$$\alpha := \sum_{n=0}^{\infty} a_n$$

$\forall 0 < x < 1$  .

i.e.  $f(x) - \alpha = (1-x) \sum_{n=0}^{\infty} (\alpha_n - \alpha) x^n$  .  $\forall 0 < x < 1$  .

Let  $\varepsilon > 0$  . As  $\alpha_n \rightarrow \alpha$ ,  $\exists N \in \mathbb{N}$  s.t.

$$|\alpha_n - \alpha| < \varepsilon/2 \quad \forall n \geq N$$

$$\therefore |f(x) - \alpha| \leq |1-x| \times \sum_{n=0}^{\infty} |d_n - \alpha| x^n.$$

$\forall 0 < x < 1$

$$|1-x| = 1-x$$

$$= |1-x| \times \left\{ \sum_{n=0}^{N-1} |d_n - \alpha| x^n + \sum_{n=N}^{\infty} |d_n - \alpha| x^n \right\}$$

$$< |1-x| \times \left\{ \sum_{n=0}^{N-1} |d_n - \alpha| + \frac{\varepsilon}{2} \times \sum_{n=N}^{\infty} |d_n - \alpha| x^n \right\}$$

$\because x < 1$

$$< (1-x) \times \left\{ \sum_{n=0}^{N-1} |d_n - \alpha| + \frac{\varepsilon}{2} \times (1-x)^{-1} \right\}$$

$\because 0 < x < 1$

$$= (1-x) \times \sum_{n=0}^{N-1} |d_n - \alpha| + \varepsilon/2.$$

Now  $x \rightarrow 1^- \Rightarrow (1-x) \rightarrow 0$

Now for the same  $\varepsilon > 0$ ,  $\exists \delta > 0$  s.t.

$$1-x < \frac{\varepsilon}{2 \times \sum_{n=0}^{N-1} |d_n - \alpha|} \quad \forall 0 < 1-x < \delta.$$

$$\therefore |f(x) - \alpha| < \varepsilon \quad \forall 0 < 1-x < \delta.$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \alpha \quad \left( = \sum_{n=0}^{\infty} a_n \right)$$

Similar technique with the help of Cauchy Criterion for uniform convergence  $\Rightarrow$ .

Thm (Abel's thm) If  $\sum a_n x^n$  conv. on  $(-R, R)$  & if  $\sum_{n=0}^{\infty} a_n R^n$  Converges, then  $\sum_{n=0}^{\infty} a_n x^n$  is uniformly converges on  $[0, R]$ .

HW

# Recall:  $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$ ,  $|x| < 1$ .

By integ. term-by-term

$$\Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\forall x \in (-1, 1).$$

i.e.  $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$  on  $(-1, 1)$ . ← we already know this.

∴  $1 - \frac{1}{2} + \frac{1}{3} - \dots$  conv. by Abel's thm.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots$$

∴

This is an exciting equality!!

i.e. Alt-harmonic series =  $\ln 2$ !!



## Double Sequences

Def: A ~~real valued~~ fn.  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$  (or  $f: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ ) is called a double seqn.

We write  $f$  simply as  $\{f(m, n)\}$  or  $\{a_{m, n}\}_{m, n \in \mathbb{N}}$ .  
 $\downarrow$   
 $a_{m, n} := f(m, n)$   
 $\forall (m, n) \in \mathbb{N} \times \mathbb{N}$ .

eg:  $\left\{ \frac{1}{m+n} \right\}_{m, n \in \mathbb{N}}$ ,  $\left\{ e^{mn} \right\}_{m, n \in \mathbb{N}}$ ,  $\left\{ m + \cos mn \right\}_{m, n \in \mathbb{Z}_+}$

Obs: For each fixed  $m \in \mathbb{N}$ ,  $\{a_{m, n}\}_{n=1}^{\infty}$  is a Seqn..

$\forall$   $n \in \mathbb{N}$ ,  $\{a_{m, n}\}_{m=1}^{\infty}$   $\parallel$  \_\_\_\_\_.

$\therefore$  It make sense to talk about:

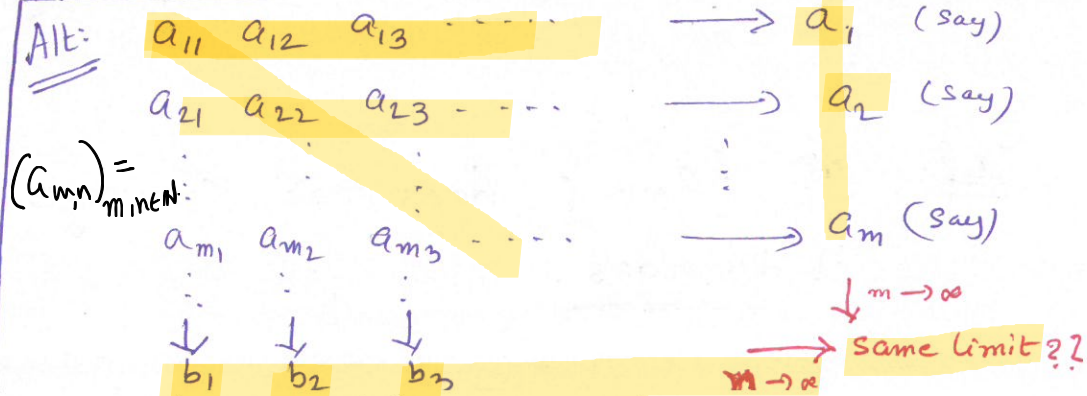
$$\lim_{n \rightarrow \infty} \left( \lim_{m \rightarrow \infty} a_{mn} \right) \quad \& \quad \lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} a_{mn} \right)$$

Q:

$\downarrow$  ??  $\downarrow$

$$a_m = \lim_{n \rightarrow \infty} a_{m, n}$$

Ans: No.



eg:  $a_{m,n} = \frac{n}{m+n} \quad \forall m, n \geq 1.$

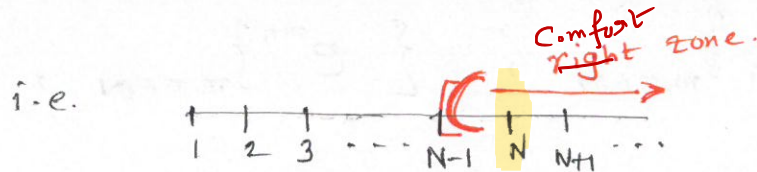
$\therefore \lim_{n \rightarrow \infty} a_{m,n} = 1 \neq 0 = \lim_{m \rightarrow \infty} a_{m,n}.$

$\uparrow$   $\times m.$                        $\uparrow$   $\times n.$

$\therefore \lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} a_{m,n} \right) \neq \lim_{n \rightarrow \infty} \left( \lim_{m \rightarrow \infty} a_{m,n} \right)$  in general.

Q: How to define convergence of  $\{a_{m,n}\}$ ?

For  $\{a_n\}$ , we say  $a_n \rightarrow a$  if for  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.  $|a_n - a| < \varepsilon \quad \forall n \geq N.$

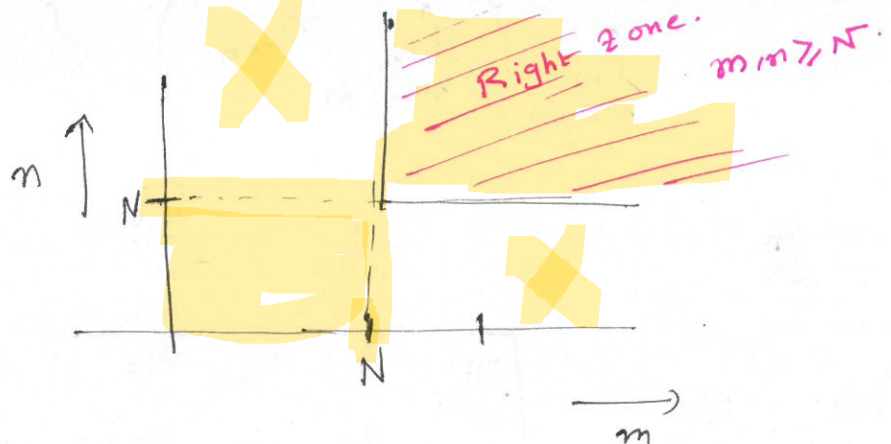


Similarly

Def: A double seqn.  $\{a_{m,n}\}$  converges to the double limit  $a$  if for  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$$|a_{m,n} - a| < \varepsilon \quad \forall m, n \geq N.$$

HW:  $a$  is!



Def: If  $\{a_{m,n}\}$  does not converge, we say that it diverges.



Def: Iterated limits of the double seqn  $\{a_{m,n}\}$  are:

$$\lim_{n \rightarrow \infty} \left( \lim_{m \rightarrow \infty} a_{m,n} \right) = \& \lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} a_{m,n} \right).$$

*double limit*

Thm: Let  $a_{m,n} \rightarrow a$  as  $m, n \rightarrow \infty$ . If  $\lim_{n \rightarrow \infty} a_{m,n}$  exists  $\forall m$ , then  $\lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} a_{m,n} \right) = a$ .

[Why if  $\lim_{m \rightarrow \infty} a_{m,n}$  exists  $\forall n$ , then  $\lim_{n \rightarrow \infty} \left( \lim_{m \rightarrow \infty} a_{m,n} \right) = a$ ]

Proof: Set  $\alpha_m := \lim_{n \rightarrow \infty} a_{m,n} \quad \forall m$ . [Claim:  $\alpha_m \rightarrow a$ ]

Fix  $\varepsilon > 0$ .  $\exists N \in \mathbb{N}$  s.t.

$$|a_{m,n} - a| < \varepsilon/2 \quad \forall m, n \geq N.$$

$\therefore \lim_{n \rightarrow \infty} a_{m,n} = \alpha_m$  ~~for all m~~, for each  $m \in \mathbb{N}$   
 $\uparrow$   
for all m

$\exists N(m) \in \mathbb{N}$  s.t.

$$|a_{m,n} - \alpha_m| < \varepsilon/2 \quad \forall n \geq N(m).$$

Fix  $m \geq N$ . Then pick  $n \in \mathbb{N}$  s.t.  $n \geq N(m)$  &  $n \geq N$ .

$$\therefore |\alpha_m - a| \leq |\alpha_m - a_{m,n}| + |a_{m,n} - a|$$

$$< \varepsilon/2 + \varepsilon/2. \quad \forall$$

$$\Rightarrow |\alpha_m - a| < \varepsilon \quad \forall m \geq N.$$

$$\Rightarrow \alpha_m \rightarrow a.$$

$\square$

$\{(-1)^n (\frac{1}{2} + \frac{1}{n})\}$   $\frac{1}{m}, \frac{1}{n} \rightarrow 0$   $\Rightarrow \frac{1}{m} + \frac{1}{n} \rightarrow 0$  as  $m, n \rightarrow \infty$   $|a_{m,n} - 0| = |\frac{1}{m} + \frac{1}{n}|$  (78)

eg: " $\Leftarrow$ " i.e.  $\lim_{m \rightarrow \infty} (\lim_{n \rightarrow \infty} a_{m,n})$  exists  $\nRightarrow \lim_{m, n \rightarrow \infty} a_{m,n}$  exists.

$\# a_{m,n} := (-1)^{m+n} (\frac{1}{m} + \frac{1}{n}) \Rightarrow |a_{m,n}| = |\frac{1}{m} + \frac{1}{n}| \Rightarrow a_{m,n} \rightarrow 0$  as  $m, n \rightarrow \infty$

However, for each  $m \in \mathbb{N}$ ,  $\{a_{m,n}\}$  div. & also for each  $n \in \mathbb{N}$ ,  $\{a_{m,n}\}$  div.  $\Rightarrow$  iterated limit DNE!!

Thm: (Cauchy Criterion)  $\{a_{m,n}\}$  converges  $\Leftrightarrow$  for  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$\{a_{m,n}\}$  converges  $\Leftrightarrow$  for  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$|a_{m,n} - a_{p,q}| < \varepsilon$

$\forall m, p \geq N$   
 $\& n, q \geq N$

$m, p \geq N$   
 $n, q \geq N$

Stolz  
 (Stolz angle)  
 $\{a_n\}$  conv.  $\Leftrightarrow |a_n - a_{n-1}| < \varepsilon$   $\& n \geq N$

Proof: " $\Rightarrow$ " Let  $a_{m,n} \rightarrow a$  as  $m, n \rightarrow \infty$

Let  $\varepsilon > 0$ . Then  $\exists N \in \mathbb{N}$  s.t.

$|a_{m,n} - a| < \varepsilon/2 \quad \forall m, n \geq N$

Comparison test  
 $0 \leq a_{m,n} \leq b_{m,n}$   
 $\& m, n \geq N$   
 If  $\{b_{m,n}\}$  conv.  
 then  $\{a_{m,n}\}$  also conv.

For  $m \geq p \geq N$  &  $n \geq q \geq N$ ,

$|a_{m,n} - a_{p,q}| \leq |a_{m,n} - a| + |a_{p,q} - a|$   
 $< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\Leftarrow$  Let  $\varepsilon > 0$ . Then  $\exists N \in \mathbb{N}$  s.t.  $\otimes$  holds.

$\forall n \in \mathbb{N}$ , set  $d_n := a_{n,n}$   $\leftarrow$  "the diagonal."

$\therefore \otimes \Rightarrow |d_n - d_p| < \varepsilon \quad \forall n \geq p \geq N$

$\Rightarrow \{d_n\} \subseteq \mathbb{R}$  is Cauchy.

$\Rightarrow \{d_n\}$  converges. Let  $d_n \rightarrow a$  as  $n \rightarrow \infty$ .



$\therefore$  For  $\varepsilon > 0 \exists N_0 \in \mathbb{N}$  s.t.

$$|x_n - a| < \varepsilon/2 \quad \forall n \geq N_0$$

Set  $\tilde{N} := \max\{N, N_0\}$ .

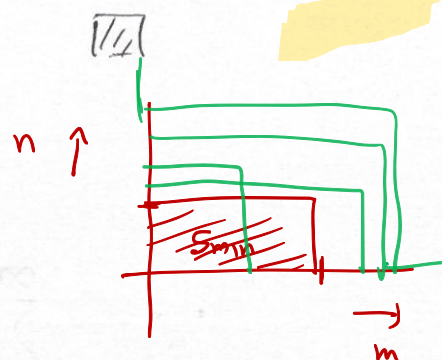
"Clever trick"

$\therefore \forall m, n \geq \tilde{N}$ , we have:

$$|a_{m,n} - a| \leq |a_{m,n} - \underbrace{a_{n,n}}_{a_n}| + |\underbrace{a_{n,n}}_{a_n} - a|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$$\Rightarrow a_{m,n} \rightarrow a.$$



### § Double Series:

Given a double seqn.  $\{a_{m,n}\}$ , we set

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n a_{i,j} \quad \forall m, n \geq 1.$$

(m,n)-th partial sum.

Given  $\{a_{m,n}\}$ , the formal sum  $\sum_{m,n=1}^{\infty} a_{m,n}$  is called double series.

The double seqn.  $\{S_{m,n}\}$  is said to be the double series generated by  $\{a_{m,n}\}$ , & denoted by  $\sum_{m,n=1}^{\infty} a_{m,n}$ .

# If  $\lim_{m,n \rightarrow \infty} S_{m,n} = a$  (i.e. Converges),

then we say that  $\sum_{m,n=1}^{\infty} a_{m,n}$  converges & write

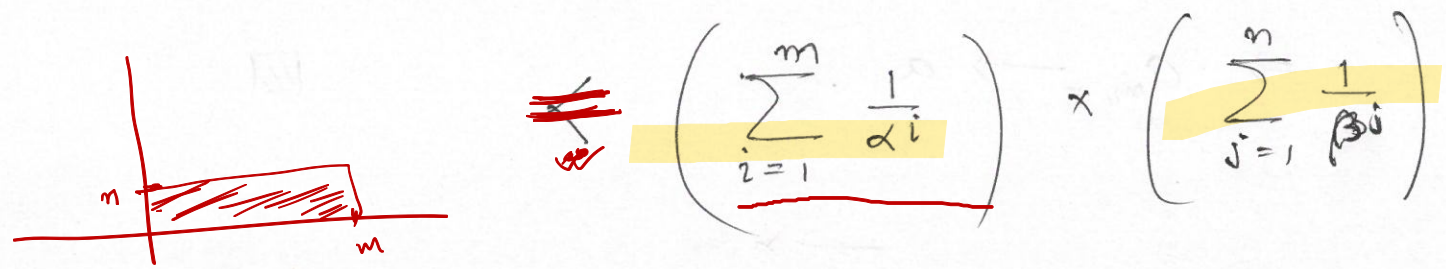
$$\sum_{m,n=1}^{\infty} a_{m,n} = a.$$

HW: Let  $\sum_{m,n=1}^{\infty} a_{m,n}$  Converges. Then  $a_{m,n} \rightarrow 0$  as  $m,n \rightarrow \infty$ .

HW: Let  $a_{m,n} \geq 0 \quad \forall m,n$ . Then  $\sum_{m,n=1}^{\infty} a_{m,n}$  Converges  $\Leftrightarrow$  the double seq.  $\{s_{m,n}\}_{m,n \geq 1}$  is bounded.

eg: Let  $\alpha, \beta > 1$ . Then  $\sum_{m,n=1}^{\infty} \frac{1}{\alpha^m \beta^n}$  is Convergent.

Proof:  $s_{m,n} = \sum_{i=1}^m \sum_{j=1}^n a_{ij} = \sum_{i=1}^m \sum_{j=1}^n \frac{1}{\alpha^i \beta^j}$  [ $a_{ij} > 0$ ]



$$\left( \sum_{i=1}^m \frac{1}{\alpha^i} \right) \times \left( \sum_{j=1}^n \frac{1}{\beta^j} \right)$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{\alpha^n}$  &  $\sum_{n=1}^{\infty} \frac{1}{\beta^n}$  are Convergent, ↖  $\alpha, \beta > 1$

$\left\{ \sum_{i=1}^m \frac{1}{\alpha^i} \right\}_{m \geq 1}$  is a bdd seq.

llly  $\left\{ \sum_{i=1}^n \frac{1}{\beta^i} \right\}_{n \geq 1}$  is a bdd seq.

$\Rightarrow \{s_{m,n}\}_{m,n \geq 1}$  is a bdd seq.

$\Rightarrow \sum_{m,n=1}^{\infty} \frac{1}{\alpha^m \beta^n}$  Converges.



Comparison test:

Let  $\sum a_{m,n}$  &  $\sum b_{m,n}$  be two double series. Suppose

$a_{m,n}, b_{m,n} \geq 0 \quad \forall m, n$  & also let

$$a_{m,n} \leq b_{m,n} \quad \forall m, n.$$

If  $\sum b_{m,n}$  Conv. then  $\sum a_{m,n}$  Conv.

— HPW —

The pending result -

you will encounter this in measure theory.

Thm: (Fubini - Tonelli theorem for series).

A double series  $\sum_{m,n=1}^{\infty} a_{m,n}$  is absolutely convergent

$\iff$  one ( & hence, both ) of the following conditions hold:

$$(i) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty,$$

$$(ii) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{m,n}| < \infty.$$

(i.e.  $\sum_{m,n=1}^{\infty} |a_{m,n}|$  exists.)

Moreover, in this case:

$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}.$$

Proof:

As usual, set

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n a_{ij}$$

&

$$\sigma_{m,n} = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|.$$

" $\Leftarrow$ " is now obvious!

$\forall m, n \geq 1$   
 $\Leftarrow$  OR, wait till the end.

$$\Rightarrow \text{Let } \sum_{m,n=1}^{\infty} |a_{m,n}| < \infty.$$

$$\textcircled{\text{m} > \text{p} \atop \text{n} > \text{q}}$$

Let  $\varepsilon > 0$ .

By Cauchy criterion:  $\exists N \in \mathbb{N}$  s.t.

$$|r_{m,n} - r_{p,q}| < \varepsilon \quad \forall \quad \left. \begin{array}{l} m \geq p \geq N \\ n \geq q \geq N \end{array} \right\}$$

Now  $|s_{m,n} - s_{p,q}| \leq |r_{m,n} - r_{p,q}| < \varepsilon \quad \forall \quad -$

$\therefore$  By Cauchy criterion, again,  $\sum a_{m,n}$  converges.

Set:  $a := \sum_{m,n=1}^{\infty} a_{m,n}$

Also, set  $r := \sup_{m,n} r_{m,n}$

$$\left[ \begin{array}{l} \sum |a_{m,n}| < \infty \text{ \& } \\ r_{m,n} = \sum_{i=1}^m \sum_{j=1}^n |a_{i,j}| \end{array} \right]$$

$$\therefore \forall \left. \begin{array}{l} i \in \mathbb{N} \\ n \in \mathbb{N} \end{array} \right\}, \quad \sum_{j=1}^n |a_{i,j}| \leq r_{i,n} \leq r.$$

$$\Rightarrow \forall i \in \mathbb{N}, \quad \sum_{n=1}^{\infty} |a_{i,n}| < \infty \Rightarrow \sum_{n=1}^{\infty} a_{i,n} \text{ converges} \quad \forall i \in \mathbb{N}.$$

$\forall m \in \mathbb{N}$ , Set  $\alpha_m := \sum_{i=1}^m \sum_{j=1}^{\infty} a_{i,j}$ .

$\therefore a = \sum_{m,n=1}^{\infty} a_{m,n}$ , for  $\varepsilon > 0 \exists N \in \mathbb{N}$  s.t.

$$|s_{m,n} - a| < \varepsilon \quad \forall m,n \geq N.$$

i.e.  $\left| \sum_{i=1}^m \sum_{j=1}^n a_{i,j} - a \right| < \varepsilon \quad \forall m,n \geq N.$



Fix  $m$  & let  $n \rightarrow \infty$ .  $\Rightarrow$

$$|a_m - a| \leq \varepsilon \quad \forall m \geq N.$$

$$\Rightarrow a_m \rightarrow a \quad \text{as } m \rightarrow \infty.$$

$$\therefore \sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}.$$

$$\text{Hwy } \sum_{m,n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}.$$

Finally (the pending <sup>issue</sup> ~~case~~):

$$\text{Let } r := \sum_{m,n=1}^{\infty} |a_{m,n}|.$$

$$r_{m,n} = \sum_{i=1}^m \sum_{j=1}^n |a_{i,j}|.$$

$$\therefore r_{m,n} \leq r \quad \forall m, n \geq 1.$$

Also, for each  $m \in \mathbb{N}$ , the seqn.  $\{r_{m,n}\}_{n=1}^{\infty}$   $\uparrow$ .

$$\Rightarrow \lim_{n \rightarrow \infty} r_{m,n} \leq r \quad \forall m \in \mathbb{N}.$$

Also, observe that  $r_{m,n} \leq r_{p,q} \quad \forall \left. \begin{matrix} m \leq p \\ n \leq q \end{matrix} \right\}$ .

$$\Rightarrow \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} r_{m,n} < \infty$$

$$\text{i.e. } \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty.$$

□



Weierstrass approximation theorem.

(A very striking result)

Q: Suppose  $f \in C[a, b]$  (we will consider  $[a, b] = [0, 1]$ : ~~no~~ No loss of generality at all). Can we "approximate"  $f$  by a polynomial  $p \in \mathbb{R}[x]$ ?

Classification/  
Ans/  
issues

Here

"approximate" means uniform ~~metric~~ <sup>norm.</sup>  $(C[a, b], d_{\text{sup}})$

i.e. "Given  $\epsilon > 0 \exists p \in \mathbb{R}[x]$  s.t.

$$\|f - p\| < \epsilon$$

$\| \cdot \|$

i.e.  $\sup_{x \in [0, 1]} |f(x) - p(x)| < \epsilon$

$\Leftrightarrow$  Given  $f \in C[a, b]$   
 $\exists \{p_n\} \subseteq \mathbb{R}[x] \rightarrow f$   
 $p_n \xrightarrow{u} f$   
 $\downarrow$  on  $[0, 1]$

The answer is yes: By 1) Weierstrass (1885). & then also

2) Bernstein (1911)  $\leftarrow$  For us.

3) Fejér (1900)  $\leftarrow$  perhaps more effective: it comes from Fourier series point of view

4) Stone (1937): More powerful result: replaces  $C[0, 1]$  by  $C(X)$   
 $\uparrow$   
Compact metric space.

$[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$

$\forall u$

# Suppose (in addition),  $f$  is  $C^\infty$ -fn. (or  $C^k$  fn.)

We can appeal to Taylor's polynomial (or even power series) approach. But it is fairly weak approximation.

Notably: i) Taylor approximation is (super) limited to points near a given point, ii) for  $n$ -degree poly. approximation, we must know/play with bound of  $(n+1)$ -th derivative, & finally what worse,  $\exists f \in C^\infty(\mathbb{R})$  [namely:  $f(x) = e^{-1/x^2}$  if  $x \neq 0$  &  $f(0) = 0$ ]

s.t.  $f^{(n)}(0) = 0 \quad \forall n \geq 0, 1, \dots$

i.e: Taylor's (or power series) approach could be completely misleading !!

— okay — So:

Thm: (Weierstrass approximation thm).

Let  $f \in C[0, 1]$ . Then  $\exists \{p_n\} \subseteq \mathbb{R}[x] \rightarrow f$  (unif.).  $(\Leftrightarrow \text{if } \varepsilon > 0 \text{ then } \exists p \in \mathbb{R}[x] \text{ s.t. } \|f - p\| < \varepsilon.)$

Idea? Introduce "bump"  $p_n$  / polynomials !!

Okay: let's do it (through Bernstein).

Let  $n \in \mathbb{N}^+$ . We know

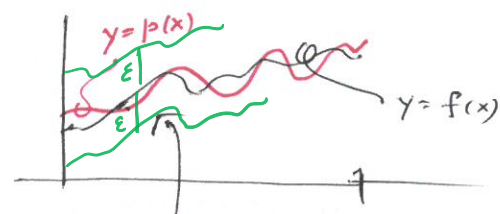
$$\sum_{k=0}^n \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{:= b_k^n} = 1$$

Def:  $b_k^n(x) := \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n, n \in \mathbb{N}.$   
Called "Bernstein polynomial".

Binomial formula:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}.$$

$a \mapsto x$   
 $b \mapsto 1-x$



do it so that the poly  $p$  remains inside the  $\varepsilon$ -band, i.e:  $f(x) - \varepsilon < p(x) < f(x) + \varepsilon \quad \forall x \in [0, 1]$

Remark: 1)  $b_k^n$  yields the necessary "bump": See through mathematica or Wikipedia picture.

2)  $\forall n \in \mathbb{N} \ \forall 0 \leq k \leq n, b_k^n$  has a ! maxima at  $x = \frac{k}{n}$ .

[See the pic. again!]

3)  $\sum_{k=0}^n b_k^n \equiv 1 \quad \forall n \in \mathbb{N}.$

4)  $\deg b_k^n = n \quad \forall 0 \leq k \leq n.$

5)  $b_k^n(x) \geq 0 \quad \forall x \in [0, 1].$

We will use this.



$$6) \quad b_k^n(1-x) = b_{n-k}^n(x) \quad \forall x \in [0,1]. \quad \text{easy}$$

$$7) \quad \int_0^1 b_k^n = \frac{1}{n+1}.$$

Anyway: (2) [along with many others] motivates us to define:

Def: Let  $f: [0,1] \rightarrow \mathbb{R}$  be a fn.  $\forall n \in \mathbb{N}$ , define the Bernstein polynomial  $B_n(f)$  as:

$$(B_n f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) b_k^n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Remark:

$$1) \quad B_n: C[0,1] \rightarrow \mathbb{R}[x].$$

$$f \mapsto B_n f \quad \leftarrow \text{a poly. of degree at most } n.$$

$$2) \quad B_n \text{ is linear: } B_n(af + g) = a B_n f + B_n g \quad \forall a \in \mathbb{R}, f, g \in C[0,1].$$

$$3) \quad \text{Let } \underbrace{f \geq g}_{\text{i.e. } f(x) \geq g(x) \forall x} \text{ in } C[0,1]. \text{ Then } \underline{B_n(f) \geq B_n(g)}. \quad \leftarrow B_n \text{ is monotonic}$$

[Indeed, enough to prove:  $B_n(f) \geq 0$  if  $f(x) \geq 0 \forall x$ .  
Straightaway follows from (5) &  $f\left(\frac{k}{n}\right) \geq 0$ .]

$$4) \quad \underline{|B_n f| \leq B_n g} \quad \text{if } |f| \leq g. \quad \leftarrow \text{we need this.}$$

$$[|f| \leq g \Leftrightarrow -g \leq f \leq g. \text{ Next: apply (3)}]$$

$$5) \quad \underline{B_n 1 = 1} \quad [\text{by (3)}].$$

$$1(x) = 1 \quad \forall x \in [0,1].$$

$$6) \quad \text{Let } f(x) = x \quad \forall x. \text{ Then } B_n f = f \quad (\text{i.e. } \underline{B_n x = x}).$$

$$\begin{aligned} B_n f &= \sum_{k=0}^n \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} \\ &= \frac{1}{n} x \sum_{k=0}^n k \binom{n}{k} x^{k-1} (1-x)^{n-k} = x \end{aligned}$$

$B_n 1, B_n x$   
 $B_n x^2$

Why? [Hint: Use  $\frac{d}{da} (a+b)^n = n(a+b)^{n-1}$ ]

7) Use

$$\Rightarrow n(a+b)^{n-1} = \sum_{k=0}^n k \binom{n}{k} a^{k-1} b^{n-k} \Rightarrow n a (a+b)^{n-1} = \sum_{k=0}^n k \binom{n}{k} a^{k-1} b^{n-k}$$

again, diff., & get:

VERY INTERESTING.

$$B_n x^2 = x^2 + \frac{x-x^2}{n}$$

You can go on like this.

[We need  $\{B_1, B_x, B_{x^2}\}$ , & some basic properties (as remarked earlier).]

Proof of Weierstrass approx. thm.

Let  $f \in C[0,1]$ ,  $\varepsilon > 0$ .  $\therefore f$  is unif. cont.  $\exists \delta > 0$  s.t.

$$|f(x) - f(y)| < \varepsilon/2 \quad \forall \quad x, y \in [0,1], \quad |x-y| < \delta$$

Set  $M := \sup_{x \in [0,1]} |f(x)|$ . Pick & fix  $a \in [0,1]$ .

$$|f(a) - f(a)| < \varepsilon$$

Then  $\forall x \in [0,1]$

$$|f(x) - f(a)| < \frac{\varepsilon}{2} + \frac{2M}{\delta^2} (x-a)^2$$

(\*)

Trivial.

If  $|x-a| < \delta$ , then  $|f(x) - f(a)| < \frac{\varepsilon}{2}$

If  $|x-a| \geq \delta$ , then  $|f(x) - f(a)| \leq 2M \leq 2M \frac{(x-a)^2}{\delta^2}$

Concise:  $= \frac{2M}{\delta^2} (x-a)^2 \leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2} (x-a)^2$

Then  $\forall x \in [0,1]$ ,  $\therefore B_n$  is linear.

$$|(B_n f)(x) - f(a)| = |B_n (f - f(a))(x)|$$

Constant  $f_a$

$$B_n(f(a))(x) = f(a) \cdot B_n(1)(x) = f(a) \cdot 1 = f(a)$$

$$\leq B_n \left( \frac{\varepsilon}{2} + \frac{2M}{\delta^2} (x-a)^2 \right)$$

Linearity of  $B_n$

$$= \frac{\varepsilon}{2} + \frac{2M}{\delta^2} B_n(x-a)^2$$

$$= B_n(x^2 - 2ax + a^2) = B_n(x^2) - 2aB_n(x) + B_n(a^2)$$

$$= \left( x^2 + \frac{x-x^2}{n} \right) - 2ax + a^2$$

$$= (x-a)^2 + \frac{x-x^2}{n}$$

$$= \frac{\varepsilon}{2} + \frac{2M}{\delta^2} (x-a)^2 + \frac{2M}{\delta^2} \left( \frac{x-x^2}{n} \right) \quad \forall x \in [0,1].$$

In particular.

$x=a$

$$|(B_n f)(a) - f(a)| \leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \left( \frac{a-a^2}{n} \right) \leq \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}$$

$$\therefore \max\{a-a^2 : 0 \leq a \leq 1\} = \frac{1}{4}$$



$$\Rightarrow |(B_n f)(a) - f(a)| \leq \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n} \quad \forall a \in [0, 1]$$

Choose  
sup of LHS.

$$\|B_n f - f\| \leq \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}$$

Choose  $N \geq \frac{M}{\delta^2 \varepsilon}$ . Then  $\forall n \geq N$ ,

$$\Rightarrow \frac{M}{2\delta^2 N} < \frac{\varepsilon}{2}$$

$$(B_n f)(x) = \sum_{k=0}^n f\left(\frac{x}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

$\varepsilon > 0$   $\exists p_n = B_n f$   
s.t.  $\|p_n - f\| < \varepsilon$

$$\| \underbrace{B_n f}_{p_n} - f \| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n \geq N$$

□

Thank you 😊

