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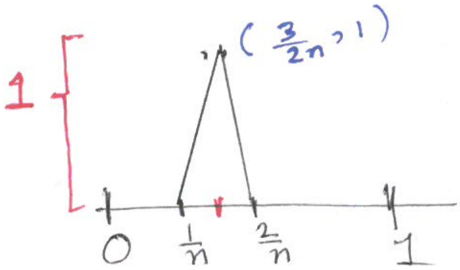
(3) f_n ~~monotonic~~ monotonicity of f_n is also necessary:

Define $f_n : [0,1] \longrightarrow \mathbb{R}$ by



$\therefore$ $f_n \in C[0,1]$ & f_n not monotone.

Also $f_n \longrightarrow 0$ pointwise but

$\|f_n\| = 1 \ \forall n \implies f_n \not\to 0$ unif.

— x —

Recall: (Dirichlet's test for convergence)

If $\sum_{n=1}^{\infty} a_n$ has bounded partial sums (i.e. $\{S_n\}$ is bdd where $S_n = \sum_{j=1}^{n} a_j \ \forall n$) & $b_n \downarrow 0$, then $\sum a_n b_n$ is convergent.

Also: Abel's test: Let $\sum a_n$ converges & $\{b_n\}$ is bdd & monotonic. Then $\sum a_n b_n$ converges.

We will reprove them: But similar technique!!

The fn' theoretic counterparts:

Thm: (Abel's test) Let $\sum f_n$ converges uniformly on S, & let $\{g_n\}$ be uniformly bdd monotone seqn of ~~real~~ fn's on S. Then

$\sum f_n g_n$ converges uniformly on S.

Proof: Set $s_n(x) := \sum_{j=1}^{n} f_j(x)$ ⟵ n-th partial sum of $\sum f_n(x)$.

$\forall x \in S$.

Then $\forall\ m > n \geq 1$, we have:

$$\sum_{j=n+1}^{m} f_j(x)\, g_j(x) = \big(s_m(x) - s_n(x)\big)\, g_{n+1}(x) + \sum_{j=n+1}^{m} \big(s_m(x) - s_j(x)\big)\big(g_{j+1}(x) - g_j(x)\big).$$

$(s_m - s_n)(x)$

$\forall x \in S$.

Abel's partial summation formula. [HW: Easy to prove.]

Let $\varepsilon > 0$. $\because \sum f_n$ Converges unif. $\exists\ N \in \mathbb{N}$ s.t.

$$\| s_m - s_n \| < \varepsilon \quad \forall\ m > n \geq N.$$

Cauchy criterion

Also, $\{g_n\}$ is uniformly bdd, $\exists\ M > 0$ s.t.

$$\| g_n \| < M \quad \forall\ n \geq 1.$$

$\therefore \forall x \in S$, & $m > n \geq N$

$$\left| \sum_{j=n+1}^{m} f_j(x)\, g_j(x) \right| \leq \left| s_m(x) - s_n(x) \right| \left| g_{n+1}(x) \right| + \sum_{j=n+1}^{m} \left| s_m(x) - s_j(x) \right| \left| g_{j+1}(x) - g_j(x) \right|$$

$$< \varepsilon \times M + \varepsilon \underbrace{\sum_{j=n+1}^{m} \left| g_{j+1}(x) - g_j(x) \right|}_{}$$

~~Needs to a fix!!~~

We need to fix this!!

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$\because \{g_n\}$ is monotonic, $\sum_{j=n+1}^{m} |g_{j+1}(x) - g_j(x)|$ is a telescoping sum, &

$$\sum_{j=n+1}^{m} |g_{j+1}(x) - g_j(x)| = |g_{n+1}(x) - g_{m+1}(x)|$$

$$\leq 2M.$$

$\therefore \forall\ m > n \geq N$ & $x \in S$, we have:

$$\left| \sum_{j=n+1}^{m} f_j(x)\, g_j(x) \right| < M \times \varepsilon + \varepsilon \times 2M.$$

$$= (3M) \times \varepsilon.$$

$$\Rightarrow \quad \left\| \sum_{j=n+1}^{m} f_j\, g_j \right\| < (3M) \times \varepsilon.$$

$$\forall\ m > n \geq N.$$

$\therefore$ By Cauchy Criterion: $\sum f_n g_n$ Converges uniformly.

$\boxed{//}$

eg: Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$.

Claim: This converges uniformly on $[0, \infty)$.

$x = 0 \Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ NOT A.C.

$\therefore$ M-test is not useful

Set $f_n(x) := \frac{(-1)^n}{n}$ $\forall$ $x \in [0,\infty)$.

(*) Constant f_n's

$\because$ $\sum \frac{(-1)^n}{n}$ is convergent, $\sum f_n$ is u.c. on $[0,\infty)$.

Next, set $g_n(x) = e^{-nx}$ $\forall$ $x \geq 0$, $n \geq 1$.

$\Rightarrow$ $\|g_n\| \leq 1$ $\forall n$ on $[0,\infty)$.

& $g_n \downarrow$.

$\therefore$ By Abel's test: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-nx}$ is u.c.

$\boxtimes$

\# $f_n \equiv$ const. occurs in "most" practical problems. (*)

eg: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} |x|^n$ is u.c. on $[-1,1]$.

$f_n(x) := \frac{(-1)^n}{n}$ on $[-1,1]$. $\forall n$.

& $g_n(x) = |x|^n$ $\forall x \in [-1,1]$ & $\forall n$.

$\therefore$ $\sum f_n$ is u.c & $g_n \downarrow$, $\|g_n\| \leq 1$.

$\therefore$ Abel's test $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} |x|^n$ is u.c. on $[-1,1]$.

$\boxtimes$

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Recall: (Abel's lemma) :
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If $\alpha \leq \sum_{j=1}^{m} \omega_j \leq \beta \quad \forall\, m = 1, \dots, n$,

Then $\forall$ decreasing $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$, we have:

$$a_1 \alpha \leq \sum_{j=1}^{n} a_j \omega_j \leq a_1 \beta .$$

Thm: (Dirichlet test for uniform convergence)

Let $\{f_n\}$, $\{g_n\}$ be sequences of fn's on S. ~~Let~~ Suppose:

(i) partial sums of $\sum f_n$ are uniformly bdd on S.

(ii) $g_n \downarrow$ S, $\cancel{g_n \geq 0}$;

(iii) $g_n \longrightarrow 0$ uniformly on S.

Then $\sum f_n g_n$ is uniformly convergent on S.

Proof: Set $s_n(x) := \sum_{j=1}^{n} f_j(x) \quad \forall\, x \in S,\ n \in \mathbb{N}$.

(i) $\Rightarrow \exists\, M > 0$ s.t. $\|s_n\| \leq M \quad \forall\, n \geq 1$.

$\Rightarrow -M \leq s_n(x) \leq M \quad \forall\, n \geq 1,\ x \in S$.

Now (ii) $\Rightarrow g_1(x) \geq g_2(x) \geq \dots \geq 0$.

$\therefore$ $\forall m > n \geqslant 1$, we have

$$\| s_m - s_n \| \leq \|s_m\| + \|s_n\| \leq 2M.$$

$$\Rightarrow \left\| \sum_{j=n+1}^{m} f_j \right\| \leq 2M.$$

$$\Rightarrow \quad -2M \leqslant \sum_{j=n+1}^{m} f_j(x) \leqslant 2M$$

$\forall\ m > n \geqslant 1$
$\&\ x \in S.$

Also, (ii) $\Rightarrow$ $g_1(x) \geqslant g_2(x) \geqslant \cdots \geqslant 0.$

$\therefore$ By Abel's lemma:

pointwise

$$-2M\, g_{n+1}(x) \leqslant \sum_{j=n+1}^{m} f_j(x) g_j(x) \leqslant 2M\, g_{n+1}(x).$$

$$\Rightarrow \left| \sum_{j=n+1}^{m} f_j(x)\, g_j(x) \right| \leq 2M\, g_{n+1}(x).$$

$\forall x \in S.$

i.e. $\displaystyle \left\| \sum_{j=n+1}^{m} f_j g_j \right\| \leqslant 2M \| g_{n+1} \|$

$\forall\ m > n \geqslant 1.$

Let $\varepsilon > 0$.

$\because$ $g_n \to 0$ uniformly, $\exists\ N \in \mathbb{N}$ s.t.

$$\| g_j \| < \frac{\varepsilon}{2M} \qquad \forall j \geqslant N.$$

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$\therefore\ \forall\ m > n \geq N$, we have:

$$\left\| \sum_{j=n+1}^{m} f_j\, g_j \right\| \leq 2M\, \|g_{n+1}\|$$

$$< 2M \times \frac{\varepsilon}{2M} = \varepsilon.$$

$\therefore$ By Cauchy criterion: $\sum_{j=1}^{\infty} f_j g_j$ is uniformly convergent in S. ☐

HW: $\forall\ x \in \mathbb{R}$, & $n \geq 1$, $2 \sin\frac{x}{2} \times [\cos x + \cos 2x + \cdots + \cos nx]$
$= \sin(n+\tfrac{1}{2})x - \sin x/2$

eg: Consider the series: $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx.$ ← Important example.

This series converges on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$.

Set $f_n(x) = \cos nx$ $\forall\ n$ & $x \in \mathbb{R}$.

~~Indeed~~ $|s_n(x)| = |\cos x + \cos 2x + \cdots + \cos nx|$

$$= \left| \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\frac{x}{2}}{2 \sin\frac{x}{2}} \right|$$

$\forall\ x \neq 2n\pi$

$$\leq \frac{1}{\left|\sin\frac{x}{2}\right|} \quad \forall\ n \geq 1.$$

i.e. $|S_n(x)| \leq 1/|\sin \frac{x}{2}|$ $\forall\, n \in \mathbb{N}$ & $x \neq 2n\pi$. —— ⊛

$\therefore$ For each fixed $x \in \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$,

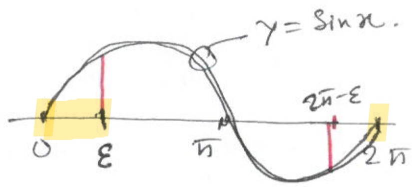
$\{S_n(x)\}$ is uniformly bdd.

Also, $\{\frac{1}{n}\} \downarrow$, & $\frac{1}{n} \to 0$.

$\therefore$ By the Dirichlet test (applied to series of real no's)

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos(nx) \text{ Converges. } \forall\, x \in \mathbb{R} \setminus \{2n\pi\}$$

-HW- Illy $\sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$ —— " —— " —— .



See this for $y = \sin \frac{x}{2}$

Something more is true:

Let $0 < \varepsilon < 2\pi$.

Then for $x \in [\varepsilon, 2\pi - \varepsilon] \longmapsto \sin \frac{x}{2}$,

the absolute minimum value is assumed at $x = \varepsilon$ or $x = 2\pi - \varepsilon$. i.e.

$$\left|\sin \frac{x}{2}\right| \geq \left|\sin \frac{\varepsilon}{2}\right| \Rightarrow \frac{1}{|\sin \frac{x}{2}|} \leq \frac{1}{|\sin \frac{\varepsilon}{2}|}$$

$\forall\, x \in [\varepsilon, 2\pi - \varepsilon]$.

$\therefore$ ⊛ $\Rightarrow$ $|S_n(x)| \leq \frac{1}{|\sin \varepsilon/2|}$ $\forall\, x \in [\varepsilon, 2\pi - \varepsilon]$ $\forall n$.

$\varepsilon \to \delta$

$[\delta, 2\pi - \delta]$

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Then, with $g_n(x) = \frac{1}{n}$, $x \in [\varepsilon, 2\pi - \varepsilon]$, we conclude by the (full) Dirichlet test, that

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx \quad \text{converges uniformly on } [\varepsilon, 2\pi - \varepsilon]$$

$\forall \; 0 < \varepsilon < 2\pi$.

$\boxed{//}$