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Double Sequences

Def: A ~~real valued~~ fn. $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ $\left(\text{or } f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R} \right)$ is called a double seqn.

We write f simply as $\{f(m,n)\}$ or $\underbrace{\{a_{m,n}\}_{m,n \in \mathbb{N}}}$

$a_{m,n} := f(m,n)$ $\forall (m,n) \in \mathbb{N} \times \mathbb{N}$.

eg: $\left\{ \frac{1}{m+n} \right\}_{m,n \in \mathbb{N}}$, $\left\{ e^{mn} \right\}_{m,n \in \mathbb{N}}$, $\left\{ m + \cos mn \right\}_{m,n \in \mathbb{Z}_+}$

Obs: For each fixed $m \in \mathbb{N}$, $\{a_{m,n}\}_{n=1}^{\infty}$ is a seqn.

Illy ——— $n \in \mathbb{N}$, $\{a_{m,n}\}_{m=1}^{\infty}$ —— " ——.

$\therefore$ It make sense to talk about:

$$\lim_{n \to \infty} \left(\lim_{m \to \infty} a_{mn} \right) \quad \& \quad \lim_{m \to \infty} \Big(\overbrace{\lim_{n \to \infty} a_{mn}}^{\substack{a_m \\ //}} \Big).$$

Q: $\quad \downarrow \qquad \overset{??}{=} \qquad \downarrow$

$a_m = \lim_{n \to \infty} a_{m,n}$.

Ans: No.

Alt: $(a_{mn})_{m,n \in \mathbb{N}} =$

$$\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & \cdots & \longrightarrow & a_1 \text{ (say)} \\
a_{21} & a_{22} & a_{23} & \cdots & \longrightarrow & a_2 \text{ (say)} \\
\vdots & & & & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & \longrightarrow & a_m \text{ (say)} \\
\vdots & \vdots & \vdots & & & \downarrow m \to \infty \\
\downarrow & \downarrow & \downarrow & & & \\
b_1 & b_2 & b_3 & & \xrightarrow[m \to \infty]{} & \text{same limit ??}
\end{array}$$

eg: $a_{m,n} = \dfrac{n}{m+n} \quad \forall\, m, n \geq 1.$

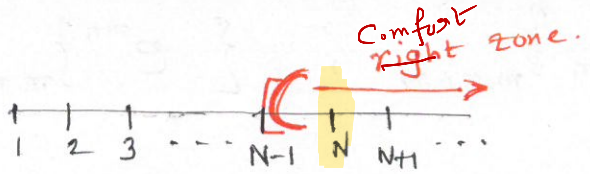
$$\therefore \lim_{n\to\infty} a_{m,n} \underset{\forall m.}{=} 1 \neq 0 \underset{\forall n.}{=} \lim_{m\to\infty} a_{m,n}.$$

$$\therefore \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right) \neq \lim_{n\to\infty}\left(\lim_{m\to\infty} a_{m,n}\right) \text{ in general.}$$

Q: How to define convergency of $\{a_{m,n}\}$?

For $\{a_n\}$, we say $a_n \to a$ if for $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon \quad \forall\, n \geq N$.

i.e.

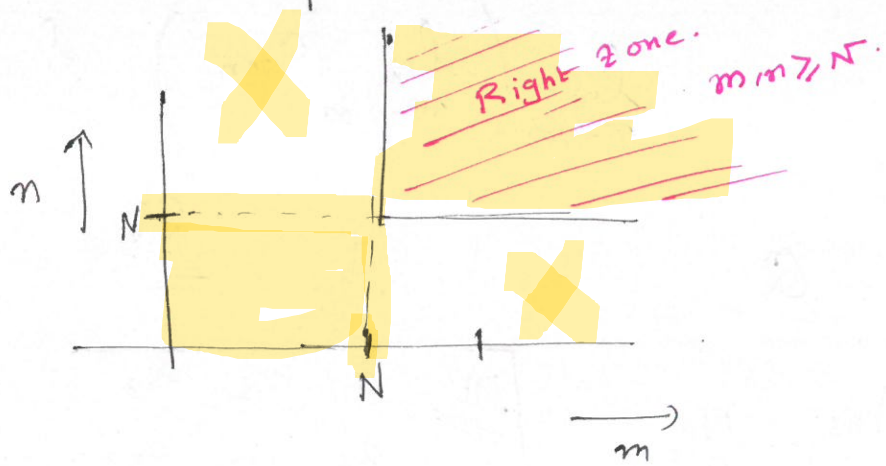


Similarly

Def: A double seqn. $\{a_{m,n}\}$ converges to the double limit a if for $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t.

$$|a_{m,n} - a| < \varepsilon \quad \forall\, m, n \geq N.$$

HW: a is !



Def: If $\{a_{m,n}\}$ does not converge, we say that it diverges.

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Def: Iterated limits of the double seq^n $\{a_{m,n}\}$ are:

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{m,n}\right) \quad \& \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right).$$

Thm: Let $a_{m,n} \longrightarrow \underbrace{a}_{\text{double limit}}$ as $m,n\to\infty$. If $\lim_{n\to\infty} a_{m,n}$ exists $\forall m$, then $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right) = a$.

[Illy if $\lim_{m\to\infty} a_{m,n}$ exists $\forall n$, then $\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{m,n}\right) = a$]

Proof: Set $\alpha_m := \lim_{n\to\infty} a_{m,n} \quad \forall m$. [Claim: $\alpha_m \to a$]

Fix $\varepsilon > 0$. $\exists\, N \in \mathbb{N}$ s.t.

$$|a_{m,n} - a| < \varepsilon/2 \quad \forall\, m,n \geq N.$$

$\because \lim_{n\to\infty} a_{m,n} = \alpha_m$ ~~for all m~~, for each $m \in \mathbb{N}$ (for all m)

$\exists\, N(m) \in \mathbb{N}$ s.t.

$$|a_{m,n} - \alpha_m| < \varepsilon/2 \quad \forall\, n \geq N(m).$$

Fix $m \geq N$. Then pick $n \in \mathbb{N}$ s.t. $n \geq N(m)$ & $n \geq N$.

$$\therefore |\alpha_m - a| \leq |\alpha_m - a_{m,n}| + |a_{m,n} - a|$$

$$< \varepsilon/2 + \varepsilon/2 .$$

$$\Rightarrow |\alpha_m - a| < \varepsilon \quad \forall\, m \geq N.$$

$$\Rightarrow \alpha_m \to a.$$

☑

$m=2$ $\left\{(-1)^n\left(\frac{1}{2}+\frac{1}{n}\right)\right\}$. $\frac{1}{m}, \frac{1}{n} \to 0 \Rightarrow \frac{1}{m}+\frac{1}{n} \to 0$ as $m,n\to\infty$ $|a_{m,n}-0| = \left|\frac{1}{m}+\frac{1}{n}\right|$

eg: "$\Leftarrow$" i.e. $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right)$ exists $\not\Rightarrow$ $\lim_{m,n\to\infty} a_{m,n}$ exists.

$a_{m,n} := (-1)^{m+n}\left(\frac{1}{m}+\frac{1}{n}\right). \Rightarrow |a_{m,n}| = \left|\frac{1}{m}+\frac{1}{n}\right| \Rightarrow a_{m,n} \to 0$ as $m,n\to\infty$

However, for each $m\in\mathbb{N}$, $\{a_{m,n}\}$ div. & also for each $n\in\mathbb{N}$, $\{a_{m,n}\}$ div. $\Rightarrow$ iterated limit DNE !!

Thm: (Cauchy Criterion)

"Stolz" ((Stolz angle))

$\{a_{m,n}\}$ Converges $\Leftrightarrow$ for $\varepsilon>0$ $\exists$ $N\in\mathbb{N}$ s.t.

$$|a_{m,n} - a_{p,q}| < \varepsilon \quad \forall\ m \geq p \geq N \ \ \& \ \ n \geq q \geq N. \qquad \otimes$$

$m, p \geq N$
$n, q \geq N$

$\{x_n\}$ Conv. $\Leftrightarrow |x_m - x_n| < \varepsilon$ $\forall\, m,n \geq N$

Proof: "$\Rightarrow$" Let $a_{m,n} \to a$. as $m,n\to\infty$

Let $\varepsilon>0$. Then $\exists$ $N\in\mathbb{N}$ s.t.

N (m,n) (p,q) N

$$|a_{m,n} - a| < \varepsilon/2 \quad \forall\ m,n \geq N.$$

Comparison test
$0 \leq a_{m,n} \leq b_{m,n}$
$\forall\ m,n \geq N$.
If $\{b_{m,n}\}$ Conv. then $\{a_{m,n}\}$ also Conv.

$\therefore$ For $m \geq p \geq N$ & $n \geq q \geq N$,

$$|a_{m,n} - a_{p,q}| \leq |a_{m,n} - a| + |a_{p,q} - a|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

"$\Leftarrow$" Let $\varepsilon>0$. Then $\exists$ $N\in\mathbb{N}$ s.t. $\circledast$ holds.

$\forall n\in\mathbb{N}$, set $\alpha_n := a_{n,n}$. $\leftarrow$ "the diagonal."

$\therefore \circledast \Rightarrow |\alpha_n - \alpha_p| < \varepsilon \quad \forall\ n \geq p \geq N$

$\Rightarrow \{\alpha_n\} \subseteq \mathbb{R}$ is Cauchy.

$\Rightarrow \{\alpha_n\}$ Converges. Let $\alpha_n \to a$. as $n\to\infty$.

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$\therefore$ For $\varepsilon > 0$ $\exists$ $N_0 \in \mathbb{N}$ s.t.

$$|\alpha_n - a| < \varepsilon/2 \quad \forall\, n \geq N_0$$

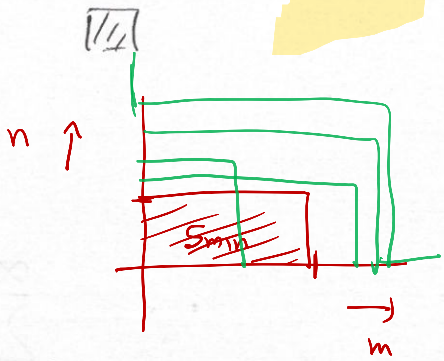
Set $\tilde{N} := \max\{N, N_0\}$. "Clever trick"

$\therefore$ $\forall$ $m, n \geq \tilde{N}$, we have:

$$|a_{m,n} - a| \leq |\underbrace{a_{m,n} - a_{n,n}}_{\alpha_n}| + |\underbrace{a_{n,n} - a}_{\alpha_n}|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\Rightarrow$ $a_{m,n} \longrightarrow a$.



— x —

§ Double Series:

Given a double seq^n $\{a_{m,n}\}$, we set

$$S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \qquad \forall\ m, n \geq 1.$$

(m,n)-th partial sum.

Given $\{a_{m,n}\}$, the formal sum $\sum_{m,n=1}^{\infty} a_{m,n}$ is call double series.

The double seq^n $\{S_{m,n}\}$ is said to be the double series generated by $\{a_{m,n}\}$, & denoted by $\sum_{m,n=1}^{\infty} a_{m,n}$.

If $\lim_{m,n \to \infty} S_{m,n} = a$ (i.e. converges),

then we say that $\sum_{m,n=1}^{\infty} a_{m,n}$ converges & write

$$\sum_{m,n=1}^{\infty} a_{m,n} = a.$$

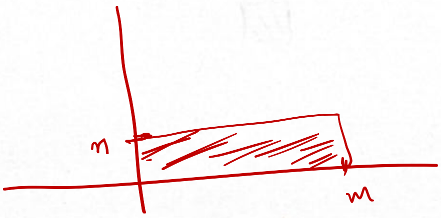
HW: Let $\sum_{m,n=1}^{\infty} a_{m,n}$ Converges. Then $a_{m,n} \longrightarrow 0$ as $m,n \longrightarrow \infty$.

~~eg:~~ **HW:** Let $a_{m,n} \geq 0 \ \forall m,n$. Then $\sum_{m,n=1}^{\infty} a_{m,n}$ Converges $\iff$ the double Seqn. $\{S_{m,n}\}_{m,n \geq 1}$ is bounded.

$\{S_{m,n}\} \uparrow$

eg: ~~Let p, q > 1. Then for each~~
Let $\alpha, \beta > 1$. Then $\sum_{m,n=1}^{\infty} \frac{1}{\alpha^m \beta^n}$ is Convergent.

Proof. $S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{\alpha^i \beta^j}$. $[a_{ij} > 0$



$$= \left(\sum_{i=1}^{m} \frac{1}{\alpha^i} \right) \times \left(\sum_{j=1}^{n} \frac{1}{\beta^j} \right).$$

$\because \sum_{n=1}^{\infty} \frac{1}{\alpha^n}$ & $\sum_{n=1}^{\infty} \frac{1}{\beta^n}$ are Convergent,

$\alpha, \beta > 1$.

$\left\{ \sum_{i=1}^{m} \frac{1}{\alpha^i} \right\}_{m \geq 1}$ is a bdd Seqn.

$|||^{ly}$ $\left\{ \sum_{i=1}^{n} \frac{1}{\beta^i} \right\}_{n \geq 1}$ is a bdd Seqn.

$\Rightarrow \{S_{m,n}\}_{m,n \geq 1}$ is a bdd Seqn

$\Rightarrow \sum_{m,n=1}^{\infty} 1/\alpha^m \beta^n$ Converges.

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Comparison test:

Let $\sum a_{m,n}$ & $\sum b_{m,n}$ be two double series. Suppose $a_{m,n}, b_{m,n} \geq 0 \ \forall\, m,n$ & also let

$$a_{m,n} \leq b_{m,n} \quad \forall\, m,n.$$

If $\sum b_{m,n}$ conv. then $\sum a_{m,n}$ conv. — HW — .

The pending result —

you will encounter this in measure theory.

Thm: (Fubini – Tonelli theorem for series).

A double series $\sum_{m,n=1}^{\infty} a_{m,n}$ is absolutely convergent

(i.e. $\lim_{m,n\to\infty} \gamma_{m,n}$ exists.)

$\iff$ one (& hence, both) of the following conditions hold:

(i) $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{m,n}| < \infty$,

(ii) $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |a_{m,n}| < \infty$.

Moreover, in this case:
$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}$$

Proof. As usual, set
$$S_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$$
&
$$\gamma_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|.$$

$\forall\, m,n \geq 1$

"$\Leftarrow$" is now obvious! ← OR, wait till the end.

$m \geq p$, $n \geq q$.

"$\Longrightarrow$" Let $\sum_{m,n=1}^{\infty} |a_{m,n}| < \infty$.

Let $\varepsilon > 0$.

By Cauchy Criterion: $\exists\ N \in \mathbb{N}$ s.t.

$$|r_{m,n} - r_{p,q}| < \varepsilon \quad \forall \left.\begin{matrix} m \geq p \geq N \\ n \geq q \geq N \end{matrix}\right\}$$

$$\left|\sum_{i=p+1}^{m} \sum_{j=q+1}^{n} a_{ij}\right| \leq \sum_{i}^{} \sum_{j=q}^{n} |a_{ij}|$$

Now $|S_{m,n} - S_{p,q}| \leq \cancel{|} r_{m,n} - r_{p,q} \cancel{|} < \varepsilon \quad \forall$ — $\forall$ —

$\therefore$ By Cauchy criterion, again, $\sum a_{m,n}$ converges.

Set: $a := \sum_{m,n=1}^{\infty} a_{m,n}$.

Also, set $r := \sup_{m,n} r_{m,n}$ $\Big[\sum |a_{m,n}| < \infty$ & $r_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|\Big)$

$\Downarrow$ $r_{m,n} \uparrow$.

$\therefore$ $\left.\begin{matrix} \forall\ i \in \mathbb{N}, \\ \& \ n \in \mathbb{N} \end{matrix}\right\}$, $\sum_{j=1}^{n} |a_{i,j}| \leq r_{i,n} \leq r$.

$\Rightarrow \forall\ i \in \mathbb{N}$, $\sum_{n=1}^{\infty} |a_{i,n}| < \infty \Rightarrow \sum_{n=1}^{\infty} a_{i,n}$ converges $\forall\ i \in \mathbb{N}$.

$\forall\ m \in \mathbb{N}$, Set $\alpha_m := \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{i,j}$.

$\because$ $a = \sum_{m,n=1}^{\infty} a_{m,n}$, for $\varepsilon > 0$ $\exists\ N \in \mathbb{N}$ s.t.

$$|S_{m,n} - a| < \varepsilon \quad \forall\ m,n \geq N.$$

i.e. $\left|\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - a\right| < \varepsilon \quad \forall\ m,n \geq N$.

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Fix m & let $n \longrightarrow \infty$. $\Longrightarrow$

$$|\alpha_m - a| \leq \varepsilon \quad \forall\ m \geqslant N.$$

$$\Longrightarrow \alpha_m \longrightarrow a \quad \text{as } m \longrightarrow \infty.$$

$$\therefore \sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}.$$

Illy

$$\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}.$$

Finally (the pending ~~case~~ issue):

Let $r := \sum_{m,n=1}^{\infty} |a_{m,n}|$. $\quad r_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$.

$$\therefore \quad r_{m,n} \leqslant r \quad \forall\ m, n \geqslant 1.$$

Also, for each $m \in \mathbb{N}$, the seqn. $\{r_{m,n}\} \uparrow$.

β_m.

$$\Rightarrow \lim_{n \to \infty} r_{m,n} \leqslant r \quad \forall\ m \in \mathbb{N}.$$

$\{\beta_m\}\uparrow$

Also, observe that $\boxed{r_{m,n} \leqslant r_{p,q}}$ $\quad \forall \left.\begin{matrix} m \leq p \\ \& \ n \leq q \end{matrix}\right\}$.

$\beta_m \leqslant r$ $\Longrightarrow$

$$\therefore \lim_{m \to \infty} \lim_{n \to \infty} r_{m,n} < \infty$$

i.e. $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty$. $\quad \blacksquare$