Weierstrass approximation theorem. (A very Striking result)

Q: Suppose $f \in C[a,b]$ (we will consider $[a,b] = [0,1]$: ~~loose~~ No loss of generality at all). Can we "approximate" f by a polynomial $p \in \mathbb{R}[x]$?

$\mathbb{R}[x] \subset C[a,b]$

Clarification / Ans / issues

Here "approximate" means uniform ~~metric~~ norm. ~~$(C[a,b], d_{sup})$~~: $\|\cdot\|$

i.e. "Given $\varepsilon > 0$ $\exists$ $p \in \mathbb{R}[x]$ s.t.

$$\|f - p\|_\infty < \varepsilon$$

i.e. "$\sup_{x \in [0,1]} |f(x) - p(x)| < \varepsilon$."

$\Longleftrightarrow$ Given $f \in C[a,b]$ $\exists \{p_n\} \subseteq \mathbb{R}[x]$ $\ni$. $p_n \xrightarrow{u} f$ on $[0,1]$!"

$\|\cdot\|$

The answer is yes: By 1) Weierstrass (1885). & then also by

2) Bernstein (1911) ← For us.

3) Fejér (1900) ← perhaps more effective: it comes from Fourier series point of view.

4) Stone (1937): More powerful result: replaces $C[0,1]$ by $C(X)$ ↑ compact metric space.

$[0,1] \times [0,1] \subseteq \mathbb{R}^2$

$\|\cdot\|_\infty$

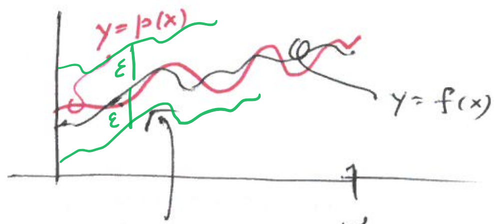
\# Suppose in addition, f is C^∞-fn. (or C^k fn.). We can appeal to Taylor's polynomial (or even power series) approach. But it is fairly weak approximation. Notably: i) Taylor approximation is (super) limited to points near a given point, ii) For n-degree poly. approximation, we must know/play with bound of $(n+1)$-th derivative, & iii) finally what worse, $\exists f \in C^\infty(\mathbb{R})$ [namely: $f(x) = e^{-1/x^2}$ if $x \neq 0$ & $f(0) = 0$]

s.t. $f^{(n)}(0) = 0 \quad \forall\ n \geq 0, 1, \ldots$

i.e: Taylor's (or power series) approach could be completely misleading !!

— okay —. So:

Thm: (Weierstass approximation thm).
Let $f \in C[0,1]$. Then $\exists\ \{p_n\} \subseteq \mathbb{R}[x]$ $\ni$ $p_n \xrightarrow{\text{unif.}} f$. ($\Leftrightarrow$ if $\varepsilon > 0$ then $\exists\ p \in \mathbb{R}[x] \ni \|f - p\| < \varepsilon$.)



do it so that the poly p remains inside the ε-"band".
i.e: $f(x) - \varepsilon < p(x) < f(x) + \varepsilon \quad \forall\ x \in [0,1]$.

Idea? Introduce "bump" fn / polynomials !!

Okay: let's do it (through Bernstein).

Let $n \in \mathbb{N}$ & fix it. We know

$$\sum_{k=0}^{n} \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{:= b_k^n} = 1.$$

Binomial formula: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.
$a \longmapsto x$
$b \longmapsto 1-x$

Def: $\boxed{b_k^n(x) := \binom{n}{k} x^k (1-x)^{n-k}}$, $0 \leq k \leq n$, $n \in \mathbb{N}$.

Called "Bernstein polynomial".

Remark:
1) b_k^n yields the necessary "bump" : See through mathematica picture or Wikipedia.
2) $\forall\ n \in \mathbb{N}$ & $0 \leq k \leq n$, b_k^n has a ! maxima at $\boxed{x = \frac{k}{n}}$.
[See the pic. again.]
We will use this.
3) $\sum_{k=0}^{n} b_k^n \equiv 1 \quad \forall\ n \in \mathbb{N}$.
4) $\deg b_k^n = n \quad \forall\ 0 \leq k \leq n$.
5) $b_k^n(x) \geq 0 \quad \forall\ x \in [0,1]$.

6) $b_k^n(1-x) = b_{n-k}^n(x) \quad \forall\, x \in [0,1].$

7) $\int_0^1 b_k^n = \frac{1}{n+1}.$

easy

Anyway: (2) [alongwith many others] motivates us to define:

Def: Let $f : [0,1] \to \mathbb{R}$ be a fn. $\forall\, n \in \mathbb{N}$, define the Bernstein polynomial $B_n(f)$ as:

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) b_k^n(x) \quad \left(= \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \right).$$

Remark #:

1) $B_n : C[0,1] \longrightarrow \mathbb{R}[x].$

$f \longmapsto B_n f \quad \leftarrow$ a poly. of degree at most n.

2) B_n is linear: $\quad B_n(af+g) = a B_n f + B_n g$

$\forall\, a \in \mathbb{R},\ f, g \in C[0,1].$

3) Let $f \geq g$ in $C[0,1]$. Then $B_n(f) \geq B_n(g)$.

i.e. $f(x) \geq g(x)\ \forall x$ $\quad$ $\leftarrow$ B_n is monotonic

[Indeed, enough to prove: $B_n(f) \geq 0$ if $f(x) \geq 0\ \forall x$.

Straightaway follows from (5) & $f\left(\frac{k}{n}\right) \geq 0$].

4) $|B_n f| \leq B_n g$ if $|f| \leq g$. $\leftarrow$ we need this.

[$|f| \leq g \iff -g \leq f \leq g$. Next: apply (3)].

$B_n 1,\ B_n(x),\ B_n x^2$

5) $B_n 1 = 1$ $\quad$ [by (3)]. $\quad$ $1(x) = 1 \ \forall x \in [0,1]$.

6) Let $f(x) = x\ \forall x$. Then $B_n f = f$ (i.e. $B_n x = x$).

$$\therefore B_n f = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}.$$

$$= \frac{1}{n} \times \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} = x$$

— why? [Hint: Use $\frac{d}{da}(a+b)^n = n(a+b)^{n-1}$]

$\Rightarrow n(a+b)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} a^{k-1} b^{n-k}$

$\Rightarrow a\, n a(a+b)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} a^k \times b^{n-k}$

7) Use again, diff., & get:

$$B_n x^2 = x^2 + \frac{x-x^2}{n}.$$

You can go on like this.

VERY INTERESTING.

[We need $\{B1, Bx, Bx^2\}$, & some basic properties (as remarked earlier).]

Proof of Weierstrass approx. thm:

Let $f \in C[0,1]$, $\varepsilon > 0$. $\because f$ is unif. cont. $\exists\ \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon/2 \quad \forall\ x, y \in [0,1],\ |x-y| < \delta.$$

Set $M := \sup_{x\in[0,1]} |f(x)|$. Pick & fix $a \in [0,1]$. — $|f(a) - f(a)| < \varepsilon$

Trivial. If $|x-a| < \delta$, then $|f(x) - f(a)| < \frac{\varepsilon}{2}$; $\frac{(x-a)^2}{\delta^2} \geq 1$

If $|x-a| \geq \delta$, then $|f(x) - f(a)| \leq 2M \leq 2M.\frac{(x-a)^2}{\delta^2}$

Curious $= \frac{2M}{\delta^2}(x-a)^2 \leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2}(x-a)^2$

Then $\forall x \in [0,1]$ $|f(x) - f(a)| < \underbrace{\frac{\varepsilon}{2}}_{|x-a|<\delta} + \underbrace{\frac{2M}{\delta^2}(x-a)^2}_{|x-a|\geq\delta}$ — $(*)$

$\because B_n$ is linear.

Then $\forall\ x \in [0,1]$,

$$|(B_n f)(x) - f(a)| = \Big|\Big(B_n\big(f - \underbrace{f(a)}_{\text{Const fn}}\big)\Big)(x)\Big|$$

$B_n(f(a))(x) = f(a).\,B_n(1)(x) = f(a)$,

$(*)$ & (4) $\leq B_n\left(\frac{\varepsilon}{2} + \frac{2M}{\delta^2}(x-a)^2\right)$.

Linearity of B_n $= \frac{\varepsilon}{2} + \frac{2M}{\delta^2} B_n(x-a)^2$.

$B_n(x^2 - 2ax + a^2)$ $B_n\left(x^2 + \frac{x - x^2}{n}\right)$

$= \left(x^2 + \frac{x-x^2}{n}\right) - 2ax + a^2$

$= (x-a)^2 + \frac{x-x^2}{n}$.

$= \frac{\varepsilon}{2} + \frac{2M}{\delta^2}$

$$= \frac{\varepsilon}{2} + \frac{2M}{\delta^2}(x-a)^2 + \frac{2M}{\delta^2}\left(\frac{x-x^2}{n}\right). \quad \forall\ x \in [0,1].$$

In particular: $x=a$ $\Rightarrow$ $|(B_n f)(a) - f(a)| \leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2}\left(\frac{a-a^2}{n}\right) \leq \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}$.

[$\because \max\{a - a^2 : 0 \leq a \leq 1\} = \frac{1}{4}$]

$\Rightarrow \quad |(B_n f)(a) - f(a)| \leqslant \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}. \quad \forall\, a \in [0,1].$

$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{n}{k}\right) x^k (1-x)^{n-k} \binom{n}{k}$

Choose

sup of LHS. $\Rightarrow$ $\|B_n f - f\| \leqslant \frac{\varepsilon}{2} + \frac{M}{2\delta^2 n}.$

Choose $\underbrace{N \geqslant \frac{M}{\delta^2 \varepsilon}}$. Then $\forall\, n \geqslant N$,

$\Rightarrow \frac{M}{2\delta^2 N} < \frac{\varepsilon}{2}.$

$\varepsilon > 0 \quad \exists\, p_n = B_n f$

s.t. $\|p_n - f\| < \varepsilon$

$\|\underbrace{B_n f}_{p_n} - f\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$ $\boxed{///}$

$\forall\, n \geqslant N.$

— x —

Thank you :)