$\therefore$ We have:

$$L(f,P) \leq S(f,P) \leq U(f,P). \qquad \forall P \in \mathcal{P}[a,b].$$

"$S(f,P,T_P)$"

BUT, this depends on ~~tag~~ tag sets.!!

We predict: $f \in \mathcal{R}[a,b] \iff$ ["$\exists\ \lambda \in \mathbb{R}$ s.t. $S(f,P) \longrightarrow \lambda$ as $\|P\| \to 0$."]

Meaning?

??

Whatever it is, ~~that~~ we will be in a good situation, ~~as~~ as λ doesn't depends on tag set!!

$F : \mathcal{P}[a,b] \to \mathbb{R}$

"$F(P) \to \lambda$" as $\|P\| \to 0$

$F : X \to \mathbb{R}$.

"$F(x) \to \lambda$" as $\|x\| \to 0$ $\overset{\text{def}}{\iff}$ $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|F(x) - \lambda| < \varepsilon$ $\forall \|x\| < \delta$

$\|\cdot\| : X \to \mathbb{R}_{\geq 0}$

~~Meaning~~

~~S~~ $S : \mathcal{P}[a,b] \times \{\text{Tags}\} \to \mathbb{R}$

$G : \mathbb{R} \to \mathbb{R}$

Def: Given $f \in \mathcal{B}[a,b]$, we say that

$$\lim_{\|P\| \to 0} S(f,P) = \lambda, \quad \text{for some } \lambda \in \mathbb{R}, \text{ if}$$

for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|S(f,P) - \lambda| < \varepsilon \quad \forall P \in \mathcal{P}[a,b] \ \& \ T_P \text{ (tag set)}$$

s.t. $\|P\| < \delta$.

Dangeר: "$\forall T_P$" is a part of the definition.

Fact: The limit is !. ($!$ = unique).] $\exists$ $\to\leftarrow$

Proof: Suppose ~~$\exists$~~ $\lim_{\|P\| \to 0} S(f,P) = \lambda_1$ & $\lim_{\|P\| \to 0} S(f,P) = \lambda_2$,

for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

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If not, let $|\lambda_1 - \lambda_2| := \varepsilon > 0$.

$\therefore \exists\ \delta_1, \delta_2 > 0$ s.t.

$$|S(f,P) - \lambda_1| < \varepsilon/2 \qquad \forall\ \|P\| < \delta_1$$

$$\& \quad |S(f,P) - \lambda_2| < \varepsilon/2 \qquad \forall\ \|P\| < \delta_2 .$$

$\therefore$ For $\delta := \min\{\delta_1, \delta_2\}$, we have:

As usual, $\forall$ tags too.

$$|S(f,P) - \lambda_1| < \varepsilon/2 \ \& \ |S(f,P) - \lambda_2| < \varepsilon/2 \quad \forall\ \|P\| < \delta .$$

$$\therefore \varepsilon = |\lambda_1 - \lambda_2| \leq |S(f,P) - \lambda_1| + |S(f,P) - \lambda_2| \quad \forall\ \|P\| < \delta .$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon \Rightarrow \varepsilon < \varepsilon \longrightarrow \leftarrow .$$

$$\Rightarrow \lambda_1 = \lambda_2 . \qquad \boxed{/\!/}$$

And, the good!! :

Thm: Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff \exists\ \lambda \in \mathbb{R} \ni$

$$\lim_{\|P\| \to 0} S(f,P) = \lambda .$$

In this case, $\int_a^b f = \lambda$.

Note: (1) Again, " $S(f,P) \longrightarrow \lambda \ \ \forall\ T_P$ ".

(2) " $\Rightarrow$ " part is more remarkable & effective.

i.e. Suppose we know that $f \in \mathcal{R}[a,b]$.

Alright: Consider $f \in C[a,b]$. ← As for example.

Consider $P_n : a = x_0 < x_0 + h < x_0 + 2h < \cdots < x_0 + (n-1)h < x_0 + nh = b$.

Consider the tag set T_P as: $\{x_0 + jh\}_{j=0}^{n-1}$ or $\{x_0 + jh\}_{j=1}^{n}$

& compute $S(f, P)$.

No seqn. in the statement. But we will get into it soon.

And, in this case, $\int_a^b f = $ "$\lim_{n \to \infty} S(f, P_n)$" is simply the limit of "Newton sums"!!

"∴ School integ. is justified!!"

Proof: "$\Rightarrow$" Let $f \in \mathcal{R}[a,b]$. Suppose $\lambda := \int_a^b f$.

Fix $\varepsilon > 0$.

$\because f \in \mathcal{R}[a,b]$, $\exists\ \delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \qquad \forall\, P \in \mathcal{P}[a,b] \text{ s.t. } \|P\| < \delta. \qquad \text{— } (*)$$

Darboux Criterion.

We know that

$$L(f,P) \leq S(f,P) \leq U(f,P) \qquad \text{— } (+)$$

& $\forall\, T_P$ $\qquad \forall\, P \in \mathcal{P}[a,b]$

$\forall \|P\| < \delta$,

$(*) \Rightarrow$ Now $U(f,P) < \varepsilon + L(f,P) \leq \varepsilon + \underline{\int} f = \varepsilon + \lambda$.

& $L(f,P) > U(f,P) - \varepsilon \geq \overline{\int} f - \varepsilon = \lambda - \varepsilon$.

$\therefore (+) \Rightarrow \quad \lambda - \varepsilon < S(f,P) < \lambda + \varepsilon \qquad \forall \|P\| < \delta$.

& T_P.

$$\Rightarrow \quad |S(f,P) - \lambda| < \varepsilon \quad \text{— } " \text{ — }.$$

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$$\Rightarrow \lim_{\|P\|\to 0} S(f, P) = \lambda.$$

"$\Leftarrow$" Suppose $\lambda := \lim_{\|P\|\to 0} S(f,P)$ exists.

Let $\varepsilon > 0$. $\therefore \exists\ \delta > 0$ s.t.

$$|S(f,P) - \lambda| < \frac{\varepsilon}{3} \quad \forall\ \|P\| < \delta \ \& \ T_P.$$

ε

$$\Rightarrow \lambda - \varepsilon/3 < S(f,P) < \lambda + \frac{\varepsilon}{3} \quad \text{—}\|\text{—}. \quad \text{———} (✡)$$

Recall: $S(f,P) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1})$; where $\{\xi_j\} = T_P$

AND $\xi_j \in [x_{j-1}, x_j]$.

(i) P is fixed. T_P is NOT.

(ii) A finite sum.

(iii) f is bounded.

$\Downarrow$

By taking inf (& sup) over $\xi_j \in [x_{j-1}, x_j]$

[i.e. inf & sup over T_P]

by (✡), we have:

$$\left.\begin{aligned} \lambda - \varepsilon/3 &\leq L(f,P) \leq \lambda + \varepsilon/3 \\ \& \quad \lambda - \varepsilon/3 &\leq U(f,P) \leq \lambda + \varepsilon/3 \end{aligned}\right\} \text{———} (\ddagger)$$

$$\therefore U(f,P) - L(f,P) \leq \lambda + \varepsilon/3 - (\lambda - \varepsilon/3)$$

$$= 2\varepsilon/3$$

$\Rightarrow \mathcal{U}(f,P) - L(f,P) < \varepsilon.$

$\therefore$ Cauchy Criterion $\Rightarrow f \in \mathcal{R}[a,b].$

Finally, $(\mp) \Rightarrow$

$$\lambda - \varepsilon/3 \leqslant L(f,P) \leqslant \int_a^b f \leqslant \mathcal{U}(f,P) \leqslant \lambda + \varepsilon/3.$$

$$\Rightarrow \left| \lambda - \int_a^b f \right| < \varepsilon/3 \qquad \forall\, \varepsilon > 0 \text{ small}.$$

$$\Rightarrow \int_a^b f = \lambda.$$

$f \in \mathcal{R}[a,b] \Leftrightarrow \lim_{\|P\| \to 0} S(f,P) = \lambda$

Useful tool.

Thm: Suppose $f \in \mathcal{R}[a,b]$ & $\{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\|P_n\| \longrightarrow 0.$

Then $\lim_{n \to \infty} S(f, P_n) = \int_a^b f.$

OR $\forall$ Tag set T_{P_n}. i.e. limit is regardless of the choice of tag sets.

Not the standard limit

Proof: Let $\varepsilon > 0$. By Darboux criterion, $\exists\, \delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \qquad \forall\, \|P\| < \delta,\ P \in \mathcal{P}[a,b].$$

$\exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\|P_n\| \to 0$

$\because \|P_n\| \longrightarrow 0$, $\exists\, N_0 \in \mathbb{N}$ s.t.

$$\|P_n\| < \delta \qquad \forall\, n \geqslant N_0.$$

CC $\quad \therefore U(f,P_n) - L(f,P_n) < \varepsilon \qquad \forall\, n \geqslant N_0.$

$$\Rightarrow \left[U(f,P_n) - \overline{\int_a^b} f \right] + \left[\underline{\int_a^b} f - L(f,P_n) \right] < \varepsilon.$$

$\because \underline{\int} f = \overline{\int} f = \int f$

$\because [\ -\text{''}-\ ]$ & $[\ -\text{''}-\ ] \geqslant 0$, it follows that:

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$$0 \leq U(f, P_n) - \int_a^b f < \varepsilon$$
$$\& \quad 0 \leq \int_a^b f - L(f, P_n) < \varepsilon.$$
$\forall\ n \geq N_0.$

$$\Rightarrow \lim_{n\to\infty} U(f, P_n) = \int_a^b f$$
$$\& \lim_{n\to\infty} L(f, P_n) = \int_a^b f.$$

"Remark:

"..." part is a general fact !!

$\because f \in \mathcal{R}[a,b] \overset{\circledast}{\Rightarrow}$ "$\exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \int_a^b f$." // $\forall \{P_n\}$ s.t. $\|P_n\| \to 0$

$\circledast$ "$\Leftarrow$" is also true!!

Finally, since $U(f, P_n) \leq S(f, P_n) \leq L(f, P_n)\ \forall n$,

by the Squeeze theorem:

$$\lim_{n\to\infty} S(f, P_n) = \int_a^b f.$$

Again, regardless of ~~tag~~ tags!!

$\boxed{}$

The above result is fair & very useful !!

$\boxed{\text{"School integration" verified ~~for~~ or justified.}}$

Consider $f \in C[a,b]$ $\leftarrow$ "A School fn"

For $n \in \mathbb{N}$, consider $P_n : a = x_0 < x_1 < \cdots < x_n = b$ with

$$x_j - x_{j-1} = \frac{b-a}{n}. \quad \leftarrow \text{"School partition".}$$

$$\therefore \|P_n\| = \frac{b-a}{n} \quad \forall\ n \in \mathbb{N}.$$

$$\Rightarrow \|P_n\| \longrightarrow 0.$$

Then for any tag set $\{\xi_j\}_{j=1}^{n}$, we have:

$$\boxed{\int_a^b f = \lim_{n \to \infty} \left[\frac{b-a}{n} \sum_{j=1}^{n} f(\xi_j) \right].} \quad \text{———} \quad (*)$$

" The school time $\xi_j := a + \frac{b-a}{n}(j-1) \quad \forall\, j = 1, \dots, n.$

end points

or $\xi_j := a + \frac{b-a}{n}\, j$ "

$\Downarrow$

The precise "school integration" !!

Remark: Of course $(*)$ holds for all $f \in \mathcal{R}[a,b]$!!

∴ So far, we have the following ($\boxed{\text{Summary}}$):

Let $f \in \mathcal{B}[a,b]$. TFAE:

① $f \in \mathcal{R}[a,b]$.

② [Cauchy Criterion]: For $\varepsilon > 0$ $\exists$ $P_\varepsilon \in \mathcal{P}[a,b]$ s.t.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

③ [Darboux Criterion]: For $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \quad \forall\, P \in \mathcal{P}[a,b] \text{ with } \|P\| < \delta.$$

④ $\exists$ $\{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\lim_{n \to \infty} [U(f,P_n) - L(f,P_n)] = 0$.

[in this case: $\lim U(f,P_n) = \lim L(f,P_n) = \int_a^b f$.].

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(5) $\exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$.

both exist.

(6) $\lim_{\|P\|\to 0} S(f,P) = \lambda$ exists ($\forall$ tag set).

In this case: $\lambda = \int_a^b f$.

$\blacksquare$

Remark: Recall: $\mathcal{R}[a,b] := \{ f \in \mathcal{B}[a,b] : f \text{ is integrable.} \}$

Also recall: # $\mathcal{B}[a,b]$ is a vector space.

$\forall f, g \in \mathcal{B}[a,b]$ & $r \in \mathbb{R}$,

$f + rg \in \mathcal{B}[a,b]$

Here $(f + rg)(t) = f(t) + rg(t)$ $\forall t \in [a,b]$

~~Sup~~

$f + rg : [a,b] \to \mathbb{R}$

$\cap$

$\mathcal{B}[a,b]$? (HW)

f is bdd. if $\exists M > 0$

$|f(x)| \le M$ $\forall x \in [a,b]$

Also $\forall f, g \in \mathcal{B}[a,b]$,

$fg \in \mathcal{B}[a,b]$. $\longleftarrow$ Why?

"Algebra"

Here $(fg)(x) = f(x)g(x)$ $\forall x \in [a,b]$

Also, if $[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}$

are bounded, then $g \circ f \in \mathcal{B}[a,b]$!!

Why?

good with compositions.

$\therefore$ We can ask all the questions for $\mathcal{R}[a,b]$.

[by replacing $\mathcal{B}[a,b]$].

$\therefore$ $\mathcal{R}[a,b]$ a vector space? An "algebra"?

Closed under prod.

~~$A \times B$~~ ~~$A \cap B$~~

$\{A\} \subseteq P(X)$

$A \setminus B$, $A \cap B$, $A \cup B$

$M_n(\mathbb{R})$

$A.B$

$A(B+C) = AB + AC$

SECONDLY:

Consider ~~Suppose~~ $\mathcal{I} : \mathcal{R}[a,b] \longrightarrow \mathbb{R}$ defined by

$$\mathcal{I}(f) = \int_a^b f \quad \forall\, f \in \mathcal{R}[a,b].$$

$(\mathcal{R}[a,b], \int)$

$\Downarrow$??

We need to think about "$\mathcal{I}$", $\mathcal{R}[a,b]$ & the structure of $\mathcal{R}[a,b]$ together. Like:

"Linear".

$$\mathcal{I}(f + rg) = \mathcal{I}(f) + r\,\mathcal{I}(g) \;?$$

This is really a natural question.

"multiplicative"

$$\mathcal{I}(fg) = \mathcal{I}(f)\,\mathcal{I}(g) \;?$$

Well, no harm in asking!!

"order preserving".

If $f \leq g$ (i.e. $f(x) \leq g(x)\ \forall\, x \in [a,b]$),

then $\mathcal{I}(f) \leq \mathcal{I}(g)$?

splits?

Let $a < c < b$, & $f \in \mathcal{R}[a,b]$.

$$\overset{?}{\Longrightarrow} \quad \mathcal{I}(f) = \int_a^c f + \int_c^b f \;?$$

ETC.!!