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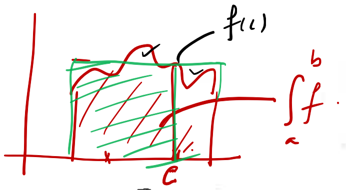
Fact: Let $f \in \mathcal{R}[a,b]$. ~~Then~~

① $m(b-a) \leq \int_a^b f \leq M(b-a)$. ← We know this.

\# ② If ~~in addition~~, $f \in C[a,b]$, then $\exists\ c \in [a,b]$ s.t.



$$f(c) = \frac{1}{b-a}\int_a^b f. \quad \longleftrightarrow \quad f(c) \times (b-a) = \int_a^b f.$$

← Kind of MVT !!

Proof: $\because$ f is cont. on $[a,b]$ (← compact), f attains all the values in $[m, M]$.

Then $m \leq \frac{1}{b-a}\int_a^b f \leq M$ ⁓ By ①

$\Longrightarrow \exists\ c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$. ▨

Def: ① $\int_a^a f := 0$ $\forall$ fn $f : S \to \mathbb{R}$, $a \in S$.

~~Fact~~ ② $\forall f \in \mathcal{B}[a,b]$, $\int_b^a f := -\int_a^b f$. $(a < b)$

Compositions

Thm: Let $f \in \mathcal{R}[a,b]$, ran $f \subseteq [c,d]$ & let $g \in C[c,d]$.

Then $g \circ f \in \mathcal{R}[a,b]$.

$$[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}, \quad g \circ f$$

[Recall: $(g \circ f)(x) = g(f(x))$ $\forall x \in [a,b]$.]

Proof: Clearly $g \circ f \in \mathcal{B}[a,b]$.

Let $\varepsilon > 0$. Since $g : [c,d] \to \mathbb{R}$ is unif. cont.

$\exists\ \delta > 0$ s.t.

$$|g(x) - g(y)| < \frac{\varepsilon}{2(b-a)} \quad \forall\ |x-y| < \delta.$$

(*)

Set $\tilde{M}_* := \sup_{y \in [c,d]} |g(y)|$.

$\because f \in \mathcal{R}[a,b]$, by Cauchy criterion, $\exists\ P \in \mathcal{P}[a,b]$

s.t. $$U(f,P) - L(f,P) < \frac{\varepsilon\delta}{4\tilde{M}_*} \quad \text{---} \quad (**)$$

Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

[Claim: $U(g\circ f, P) - L(g\circ f, P) < \varepsilon$.]

Let $J = \{1, \ldots, n\}$. & write

$$J = J_1 \coprod J_2$$

disjoint partition.

where: $J_1 = \{ j \in J : M_j - m_j < \delta \}$

& $J_2 = \{ j \in J : M_j - m_j \geq \delta \}$.

Recall: $M_j - m_j = \operatorname{osc}_{I_j} f$. $\forall j = 1, \ldots, n$.

$\because M_j - m_j = \operatorname{osc}_{I_j} f$

$= \sup_{x,y \in I_j} |f(x) - f(y)|$

$\forall j = 1, \ldots, n$,

We have: If $j \in J_1$, then $|f(x) - f(y)| < \delta \ \forall x,y \in I_j$

& by $(*)$: $|g(f(x)) - g(f(y))| < \frac{\varepsilon}{2(b-a)} \ \forall x,y \in I_j$

$$\Rightarrow \underbrace{\sup_{x,y \in I_j} |g(f(x)) - g(f(y))|}_{= \operatorname{osc}_{I_j} g\circ f} \leq \frac{\varepsilon}{2(b-a)}.$$

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For each $i = 1, \cdots, n$, we set

$$\tilde{M}_i := \sup_{I_i} g \circ f \quad \& \quad \tilde{m}_i := \inf_{I_i} g \circ f .$$

$$\therefore \ \forall j \in J_1, \quad \underbrace{\tilde{M}_j - \tilde{m}_j}_{= \operatorname{osc}_{I_j} g \circ f} \leqslant \frac{\varepsilon}{2(b-a)} .$$

$$\Rightarrow \sum_{j \in J_1} (\tilde{M}_j - \tilde{m}_j) \times |I_j| \leqslant \frac{\varepsilon}{2(b-a)} \times \sum_{j \in J_1} |I_j| .$$

$$\leqslant \frac{\varepsilon}{2(b-a)} \times (b-a).$$

$$\leqslant \varepsilon/2 .$$

All about J_1

Next, we turn to J_2:

$\tilde{M}_\varepsilon = \sup_{[a,b]} |g|$

We note that $\forall j \in J_2$, $\tilde{M}_j - \tilde{m}_j \leqslant 2 \tilde{M}_\varepsilon$

$$\therefore \sum_{j \in J_2} (\tilde{M}_j - \tilde{m}_j) \times |I_j| \leqslant 2 \tilde{M}_\varepsilon \times \sum_{j \in J_2} |I_j|$$

$$\leqslant 2 \tilde{M}_\varepsilon \times \sum_{j \in J_2} |I_j| \times \underbrace{\left(\frac{M_j - m_j}{\delta}\right)}_{\geqslant 1} .$$

$[\because \forall j \in J_2, \ M_j - m_j \geqslant \delta]$

$$= 2 \tilde{M}_\varepsilon \times \frac{1}{\delta} \times \sum_{j \in J_2} |I_j| (M_j - m_j)$$

$$\leqslant 2 \tilde{M}_\varepsilon \times \frac{1}{\delta} \sum_{j \in J} (M_j - m_j) |I_j|$$

$\because J_2 \subseteq J$

$$= \frac{2 \tilde{M}_\varepsilon}{\delta} \times \left(\mathcal{U}(f, P) - L(f, P) \right)$$

$$< \frac{2 \tilde{M}_\varepsilon}{\delta} \times \frac{\varepsilon \delta}{4 \tilde{M}_\varepsilon} \qquad (\text{by } (**))$$

$$= \frac{\varepsilon}{2} .$$

$$\therefore\ \mathcal{U}(g\circ f, P) - L(g\circ f, P) = \sum_{j\in J} (\tilde{M}_j - \tilde{m}_j)\, |I_j|$$

$$= \sum_{j\in J_1} + \sum_{j\in J_2} \quad \leftarrow \because J = J_1 \amalg J_2.$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$$\Rightarrow g\circ f \in \mathcal{R}[a,b].$$ ◪

Cor: Suppose $f\in\mathcal{R}[a,b]$. Then:

① e^f, $\sin f(x)$, $\cos f(x)$ are in $\mathcal{R}[a,b]$.

② $\sqrt[n]{f} \in \mathcal{R}[a,b]$ $\forall n\geq 1$ whenever $f(x)\geq 0$.

Q: We proved: $f\in\mathcal{R}$, $g\in C \Rightarrow g\circ f\in\mathcal{R}$.

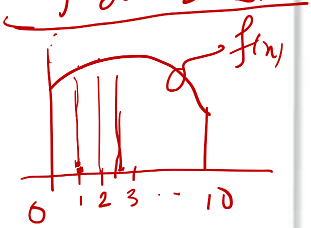
What about: "$f\in\mathcal{R}$ & $g\in\mathcal{R} \Rightarrow g\circ f\in\mathcal{R}$"?

False. Why ?? —HW—

Thm: Suppose $f, g\in\mathcal{B}[a,b]$ & ~~[illegible]~~ $f(x) = g(x)$ $\forall x\in[a,b]$ but finitely many. Then $f\in\mathcal{R}[a,b]$ $\Leftrightarrow$ $g\in\mathcal{R}[a,b]$.

Moreover, in this case, $\int_a^b f = \int_a^b g$.

$f, g\in\mathcal{B}[0,10]$



$g(x) = \begin{cases} 0 & x\in\mathbb{N} \\ f(x) & x\notin\mathbb{N} \end{cases}$

[$\therefore$ $f = 0$ on $[a,b]$ except finitely many points

$\Longrightarrow \int_a^b f = 0.$]

$\int f \neq \int g$ ✗

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Proof: Enough to assume

$$f(x) = g(x) \quad \forall\ x \in [a,b] \setminus \{c\}$$

& $f(c) \neq g(c)$.

for some $c \in [a,b]$.

The general case will then follow by induction.

So, assume the above conditions.

Let $\varepsilon > 0$ & suppose $\sup\limits_{[a,b]} |f|,\ \sup\limits_{[a,b]} |g| \leq \tilde{M}$.

for some $\tilde{M} > 0$.

[Claim: $\overline{\int} f = \overline{\int} g$ & $\underline{\int} f = \underline{\int} g$.].

For $\varepsilon > 0$ $\exists$ $P \in \mathcal{P}[a,b]$ s.t. $U(f,P) < \overline{\int} f + \varepsilon/2$. ——— ⊛

Set $\delta := \dfrac{\varepsilon}{8\tilde{M}}$.

Consider a refinement $\tilde{P}$ of P s.t. $\|\tilde{P}\| < \delta$.

Always possible.

$\because f = g$ on $[a,b]$ except $c \in [a,b]$,

f differs from g on atmost 2 subintervals of $\tilde{P}$.

[depending on c being an end point of an subinterval].

Let $\{\tilde{I}_j\}_{j=1}^{n}$ be the subintervals of $\tilde{P}$.

Assume f differs from g on I_l.

Here $l = p$, OR $l = p, p+1$, for some p.

$\Rightarrow$ $\sup\limits_{I_j} f - \sup\limits_{I_j} g = 0 \quad \forall\ j \neq p, p+1$

& $\left| \sup\limits_{I_j} f - \sup\limits_{I_j} g \right| \leq 2\tilde{M}$ for $j = p, p+1$.

The max 2-point case.

$$\therefore \quad \left| U(f,\tilde{P}) - U(g,\tilde{P}) \right|$$

$$= \left| \left(\sup_{I_p} f - \sup_{I_p} g\right) \times |I_p| + \left(\sup_{I_{p+1}} f - \sup_{I_{p+1}} g\right) \times |I_{p+1}| \right|$$

$$\leqslant 2 \times 2\, \tilde{M} \times \delta .$$

i.e. $\left| U(f,\tilde{P}) - U(g,\tilde{P}) \right| \leqslant 4\delta\tilde{M} = 4\tilde{M} \times \frac{\varepsilon}{8\tilde{M}} = \varepsilon/2 .$

i.e. $\left| U(f,\tilde{P}) - U(g,\tilde{P}) \right| \leqslant \varepsilon/2 . \longrightarrow (**)$

~~So, $\overline{\int} g \leqslant U(g,\tilde{P}) \leqslant \overline{\int} f + \varepsilon/2$ (by $(*)$)~~

Thus, $\overline{\int} g \underset{\substack{\uparrow \\ \text{always true.}}}{\leqslant} U(g,\tilde{P}) \underset{\text{by } (**)}{\leqslant} U(f,\tilde{P}) + \varepsilon/2$

$$\underset{\because \tilde{P} \supseteq P}{\leqslant} U(f,P) + \varepsilon/2 \underset{\text{by } (*)}{<} \left(\overline{\int} f + \varepsilon/2\right) + \varepsilon/2$$

i.e, $\overline{\int} g < \overline{\int} f + \varepsilon \quad \forall \varepsilon .$

$\Rightarrow \quad \overline{\int} g \leqslant \overline{\int} f .$

Switching f & g, this gives: $\quad \overline{\int} f \leq \overline{\int} g .$

$\therefore \quad \overline{\int} f = \overline{\int} g .$

$||^{ly} \quad \underline{\int} f = \underline{\int} g .$

$\therefore \quad \overline{\int} f = \underline{\int} f \iff \overline{\int} g = \underline{\int} g . \qquad \boxed{///}$