Jaydeb Sarkar.

§. The fundamental theorem of Calculus (FTC):

Note that if $\mathcal{D} :=$ diff. fn. on $\mathbb{R}$ / (a,b) / open set;

then $d : \mathcal{D} \longrightarrow \mathcal{F}(\mathbb{R})$ is a linear map.

$\mathcal{F}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \}$.

$(df := f')$.

Informal.

Illy. $\int : R[a,b] \longrightarrow \mathcal{F}(\mathbb{R})$ is a linear map. ✓

$x \mapsto \int_a^x f(t)\,dt$ ✓ $\qquad f|_{[a,x]} \in R[a,x]$

FTC ~~essen~~ essentially says:

"$d \circ \int$ = identity."

"$\int \circ d$ = identity."

TROUBLE: Compositions should ~~be~~ be well-defined first !!

In the following, we will explain the informal equalities ~~&~~ ~~then~~ & then, Along with all the necessary assumptions, we will make it more formal !!

Def: Let $S \subseteq \mathbb{R}$ & $f : S \to \mathbb{R}$ be a fn. ~~A fn~~ ~~This~~

A differentiable fn. F is called an antiderivative or a primitive of f on S if

$$f(x) = F'(x) \quad \forall\, x \in S.$$

eg: i) $\frac{1}{2}x^2$ is an antiderivative of x.

ii) $\frac{1}{2}x^2 + c$ ——— " ——— x $\forall\, c \in \mathbb{R}$.

$\therefore \quad F \xrightarrow{\frac{d}{dx}} f \xrightarrow{\frac{d}{dx}} f'$

antiderivative — function — derivative.

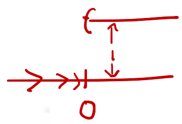
(PROVIDED: f is diff.).

Q: Do all functions have antiderivatives ??

1) polynomials. ✓.

2) Continuous fn's. ✓ ← Why ?? ← WAIT (FTC-II).

3) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ ✗ ← Why ?



Ans. "IVT"

Digression :

Thm: (Darboux's Thm)

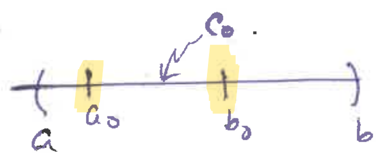
Let $f:(a,b) \longrightarrow \mathbb{R}$ be a diff. fn. & let $a < a_0 < b_0 < b$.

If $f'(a_0) < r < f'(b_0)$, then $\exists$ $c_0 \in (a_0, b_0)$ s.t.

$$f'(c_0) = r.$$

improves -sive result.

Proof:



[Note: If f' is cont. then this is straight IVT !!]

Set $g(x) := -f(x) + rx$. $x \in (a,b)$. (Here r is fixed.)

$\Rightarrow$ g diff & $\begin{cases} g'(a_0) < 0 \\ g'(b_0) > 0 \end{cases}$.

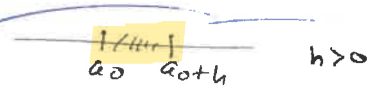
Also $g\big|_{[a_0,b_0]} : [a_0, b_0] \longrightarrow \mathbb{R}$ Cont. & hence g on $[a_0, b_0]$ is unif.

attain its extreme values.

Now $g'(a_0) > 0 \Rightarrow g(a_0+h) - g(a_0) > 0$ for $h > 0$ small.

why ??

$\Rightarrow g(a_0) < g(a_0+h)$ —— || ——.

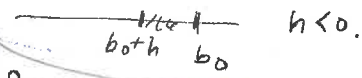


$\Rightarrow g$ does not assumes the max. at a_0.

IIIy $g'(b_0) < 0 \Rightarrow g(b_0+h) - g(b_0) > 0$ for $h < 0$ small.

why ??

$\Rightarrow g(b_0) < g(b_0+h)$ for $h<0$ small.



$\Rightarrow g$ does not assumes the max. at b_0.

(max or local max)

$\therefore g$ assumes ~~the~~ a max at $c_0 \in (a_0, b_0)$.

$\because g$ is diff. $\quad g'(c_0) = 0$.

$\Rightarrow f'(c_0) = \gamma$. ▨

Note: Thus,

$$f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

does not have an antiderivative !!

~~Alert~~ ALERT: Derivatives need not be continuous !!

$$f(x) := \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

f' is NOT cont. at 0.

THUS: Derivative of a fn need not be Cont. but still the derivative enjoys the intermediate value property!!

— the novelty of Darboux's thm.

Anyway;

The first FTC.

Thm: (FTC - I) Let $f \in \mathcal{R}[a,b]$ & $F \in C[a,b]$. Suppose F is an antiderivative of f over (a,b). Then

$$\int_a^b f = F(b) - F(a).$$

① $\therefore$ $F'(x) = f(x)$ $\forall x \in (a,b)$.

② Roughly: $\int_a^b F' = F(b) - F(a)$.

③
$$\int_{[a,b]} F' = [F]_{\partial[a,b]} = F(b) - F(a).$$

The solid interval.

boundary of $[a,b]$. $= \{a, b\}$

A remarkable result!!

④ $\partial[a,b] := \{a, b\}$

Notation

* * [In fact, we can redefine/assign F' at a & b. As $F'|_{(a,b)} = f$ & $f \in \mathcal{R}[a,b]$, the extended F' on $[a,b]$ will be integrable & $\int_a^b (\text{extended } F') = \int_a^b F'$.]

A Continuous analog of sums of differences!!

$$\text{"} \sum_{j=1}^{n} (\gamma_j - \gamma_{j-1}) = \gamma_n - \gamma_0 \text{."}$$

Proof: Let $P \in \mathcal{P}[a,b]$.

Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

$\therefore$ We have the sum of differences:

$$\sum_{j=1}^{n} \left(F(x_j) - F(x_{j-1}) \right) = F(b) - F(a).$$

Now $F\Big|_{[x_{j-1}, x_j]} \in C[x_{j-1}, x_j]$ & diff. on (x_{j-1}, x_j).

$\forall j = 1, \dots, n.$

$\therefore$ MVT $\Rightarrow$ $\exists\, \xi_j \in (x_{j-1}, x_j)$ s.t.

$$F(x_j) - F(x_{j-1}) = F'(\xi_j)\,(x_j - x_{j-1}).$$

$$\Rightarrow F(x_j) - F(x_{j-1}) = f(\xi_j)\,(x_j - x_{j-1}) \quad \text{———} \quad (*).$$

as $F'(x) = f(x)$ $\forall x$ in (a,b).

$\forall j = 1, \dots, n.$

Now $\forall j = 1, \dots, n$, we know: $\because \xi_j \in [x_{j-1}, x_j]$

$$m_j (x_j - x_{j-1}) \leq f(\xi_j)(x_j - x_{j-1}) \leq M_j (x_j - x_{j-1}).$$

$$\therefore (*) \Rightarrow \quad m_j (x_j - x_{j-1}) \leq F(x_j) - F(x_{j-1}) \leq M_j (x_j - x_{j-1}) \quad \forall j.$$

$$\Rightarrow L(f, P) \leq \sum_{j=1}^{n} \left(F(x_j) - F(x_{j-1}) \right) \leq U(f, P).$$

i.e. $L(f, P) \leq F(b) - F(a) \leq U(f, P)$ $\forall P \in \mathcal{P}[a,b]$.

$$\Rightarrow \underline{\int} f \leq F(b) - F(a) \leq \overline{\int} f.$$

But $f \in \mathcal{R}[a,b]$. $\Rightarrow$ $\displaystyle\int_a^b f = F(b) - F(a)$. ▨

$\because$ FTC–I $\Rightarrow$ $\int_a^b f$ can be computed by $\boxed{\text{finding}}$ antiderivative F of f !!

& we know, it may not be!!

Q: How to find (of course, if any!!) an antiderivative ?

— Ans: " FTC–II ".

The 2nd FTC.

Thm (FTC–II)

Let $f \in \mathcal{R}[a,b]$. Define

$$F(x) := \int_a^x f(t)\,dt \qquad \forall\ x \in [a,b].$$

Then: (1) $F \in C[a,b]$.

(2) If f is cont. at $x_0 \in (a,b)$, then F is diff. at x_0 & $F'(x_0) = f(x_0)$.

(3) If f is cont. from the right at a, then $F'_+(a) = f(a)$. $\quad$ Illy cont. from left at b.

Remember? Integration makes fn smoother !!

Proof: ~~Set Recall~~ Set $M := \sup_{x \in [a,b]} |f(x)|$. Let $\varepsilon > 0$ & $x, y \in [a,b]$.

$$\therefore\ |F(x) - F(y)| = \left| \int_x^y f(t)\,dt \right|$$

$\because$ $-M \leqslant f(t) \leqslant M \quad \forall\ t \in [a,b]$.

$\left[\ \times \Rightarrow \cancel{-M(y-x) \leqslant (y-x) f(t) \leqslant (y-x) M} \quad \forall\ t \in [a,b]\ \right]$ [assuming $y > x$]

$\because$ the constant fn's $t \mapsto \pm M$ ~~$(y-x)$~~ are integrable

We have:

$$-M(y-x) \leq \int_x^y f(t)\,dt \leq M(y-x).$$

$$\Rightarrow \left| \int_x^y f(t)\,dt \right| \leq M(y-x). \qquad [\text{For } \underline{y>x}].$$

$$\therefore \forall\ x, y \in [a,b], \quad \left| \int_x^y f(t)\,dt \right| \leq M|x-y|.$$

i.e. $$|F(x)-F(y)| \leq M|x-y| \quad \forall\ x, y \in [a,b].$$

$\Rightarrow$ F is uniformly ~~Cont~~. Cont. on $[a,b]$,

— This proves (1).

Now, for (2), let f is Cont. at $x_0 \in (a,b)$.

$$\therefore \frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt. \qquad \forall\ x \neq x_0.$$

Also, $$f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)\,dt.$$

$$\therefore \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x} [f(t)-f(x_0)]\,dt. \quad \forall\ x \neq x_0$$

Now for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $|f(t)-f(x_0)| < \varepsilon$

$\forall\ |t-x_0| < \delta$. ← by Cont. of f at x_0.

Then, $$\left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| = \frac{1}{|x-x_0|}\left| \int_{x_0}^{x} [f(t)-f(x_0)]\,dt \right|$$

$$\leq \frac{1}{|x-x_0|} \times \int_{x_0}^{x} |f(t)-f(x_0)|\,dt.$$

$x_0-\delta < t < x_0+\delta$ ← x

$$< \frac{1}{|x-x_0|} \times \varepsilon \times |x-x_0| \qquad \forall\ |x-x_0| < \delta.$$

$$\Longrightarrow \quad \left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| < \varepsilon \qquad \forall\ |x-x_0| < \delta.$$

$\Longrightarrow$ F is diff. at x_0 & $F'(x_0) = f(x_0)$. ▨

Cor: Let $f \in C[a,b]$. Then

$$\frac{d}{dx}\left(\underbrace{\int_a^x f(t)\,dt}_{\text{differentiable.}} \right) = f(x) \qquad \forall\ x \in [a,b].$$

$D \circ g$ = identity on $C[a,b]$.

$\displaystyle \text{"} \sum_{j=1}^{n} x_j - \underbrace{\sum_{j=1}^{n-1} x_j}_{\text{difference of sums!!}} = x_n \text{"}$ ⟵ Continuous analog of

Fact: ~~Continuity of f is ~~must~~ "necessary" for diff. of $x \mapsto \int_a^x f(t)\,dt$.~~

"With $f \in R[a,b]$, the Cor. may not hold." — ~~HW~~ Wi~~kedoi~~t.

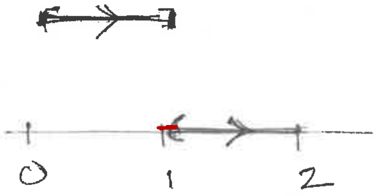
In particular, if $f \in C[a,b]$, then

$$x \longmapsto \int_a^x f(t)\,dt \quad \text{is an antiderivative of } f.$$

And of course, we know $\exists\ f \in R[a,b]$ ~~whi~~ with no antiderivatives !!

eg: $f : [0,2] \longrightarrow \mathbb{R}$ defined by :

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases}$$



Clearly, $f \in \mathcal{R}[0,2]$. $\Leftarrow$ $\because f = \chi_{[0,1]} \in \mathcal{R}[0,2]$.

Recall: if $A \subseteq B$, then $\chi_A : B \longrightarrow \mathbb{R}$ defined by

indicator or characteristic fn.

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

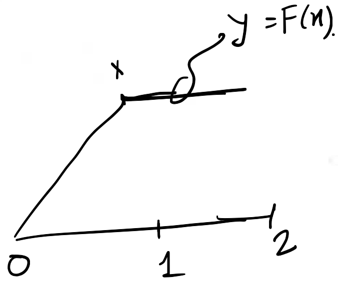
Set $F(x) := \int_0^x f(t)\,dt$. $\forall x \in [0,2]$.

If $x \in [0,1]$, then $F(x) = \int_0^x 1 \cdot dt = x$

If $x \in (1,2]$, then $F(x) = \int_0^1 f(t)\,dt + \int_1^x \underbrace{f(t)}_{=0}\,dt$.

$$= \int_0^1 1 \cdot dt + 0.$$

$$= 1 + 0 = 1.$$



$\therefore$ $$F(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x \le 2. \end{cases}$$

$\therefore$ F fails to be diff. ~~cont.~~ at $x = 1$ (precisely where f is discont.).

$\therefore$ Integration of an integrable fn. need not be diff. !!

Thm: (Integration by parts):

Let $f, g \in \mathcal{D}[a,b]$ & $f', g' \in \mathcal{R}[a,b]$. Then

$$\int_a^b f g' + \int_a^b f' g = f(b)g(b) - f(a)g(a).$$

[$f \in \mathcal{D}[a,b]$ means: $\exists$ diff. fn F on $(a-\varepsilon, b+\varepsilon)$ s.t. $F|_{[a,b]} = f$.

OR f is diff. on (a,b) & f' has an extension to $[a,b]$;

(So, extension to only 2 points: a & b).]

Negligible issue!!

Proof: Set $u = fg$. $\Rightarrow u' = f'g + fg'$.

FTC $\Rightarrow \int_a^b u' = u(b) - u(a)$.

$$\Rightarrow \int_a^b f g' + \int_a^b f' g = (fg)(b) - (fg)(a). \quad \square$$

[$\therefore \int_a^b f g' = [fg]_a^b - \int_a^b f' g$. $\Leftarrow$ the popular form!!]

Thm: (Change of variable): Let $I \subseteq \mathbb{R}$ be an open interval, $g : I \longrightarrow \mathbb{R}$ diff. & $g' \in \mathcal{R}(C)$ $\forall$ closed interval $C \subseteq I$.

Set $J = g(I)$. $\leftarrow$ (also an interval as g cont.)

If $f : J \longrightarrow \mathbb{R}$ is cont. & $a < b$ in I, then

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(x)\, dx.}$$

Proof: $\because f \circ g \in \mathcal{R}[a,b]$ (as f cont), we have that $(f \circ g)\, g' \in \mathcal{R}[a,b]$.

$\because f \in C[g(a), g(b)]$ $F(x) := \int_{g(a)}^{x} f(t)\,dt$ is diff. & $F' = f$ on ~~$[a,b]$~~ $[g(a), g(b)]$

$\mathcal{R}\,[g(b), g(a)]$

Now $(F \circ g)'(x) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$ $\forall x$ ~~$\in [a,b]$~~ $\in [g(a), g(b)]$

$\Rightarrow F \circ g' \in \mathcal{R}[a,b]$.

$$\therefore \int_a^b f(g(x))\, g'(x)\,dx = \int_a^b (F \circ g)'(x)\,dx = (F \circ g)(b) - (F \circ g)(a) = F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} F'(x)\,dx. \quad \square$$

Jaydeb Sarkar.

Change of variable

Thm: Let $u \in \mathcal{D}[a,b]$, $u' \in \mathcal{R}[a,b]$ & $f \in \mathcal{C}(u[a,b])$.

Then
$$\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f(x)\, dx.$$

The so called "u-substitution"

[Note: $[a,b] \xrightarrow{u} u[a,b] \xrightarrow{f} \mathbb{R}$]

$\Rightarrow f \circ u : [a,b] \longrightarrow \mathbb{R}$

Proof: Note that u = constant map $\Leftrightarrow u'(t) \equiv 0$.

Then the above equality is true (both sides $= 0$).

$\because u(a) = u(b)$.

So, assume that u is non constant.

$\because f$ is cont, $f \circ u \in \mathcal{R}[a,b]$. As $u' \in \mathcal{R}[a,b]$, it follows that $(f \circ u)\, u' \in \mathcal{R}[a,b]$.

Also observe that $u[a,b]$ is an interval. Closed?

$\forall x \in u[a,b]$, define $F(x) := \int_{u(a)}^{x} f(t)\, dt$.

By FTC-II, $F'(x) = f(x) \quad \forall x \in u[a,b]$.

Then ~~Also,~~ $(F \circ u)'(t) = F'(u(t))\, u'(t) = f(u(t))\, u'(t)$.

$\forall t \in [a,b]$.

$\therefore$ FTC - I $\Rightarrow$

$$\int_a^b f(u(t))\, u'(t)\, dt = \int_a^b (F\circ u)'(t)\, dt$$

$$= (F\circ u)(b) - (F\circ u)(a)$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} F'(x)\, dx.$$

$$= \int_{u(a)}^{u(b)} f(x)\, dx. \qquad \boxed{//}$$

—— x ——.