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Improper Integrals.

Assumptions for Riemann integrations:

$[a,b]$: i.e bdd & closed interval.

But we really want to integrate over (a,∞) or $(-\infty,a]$ or $(-\infty,\infty)$ etc.

$f\in\mathcal{B}[a,b]$ i.e. fn's. must be bdd.

Even a fn is unbdd., there may be some chance of finite area !!

PRACTICAL ISSUES to deal with.

Improper integrals:

$\int_a^b f$ where f is unbounded on $[a,b]$

$\int_a^\infty f$ or $\int_{-\infty}^b f$ or $\int_{-\infty}^\infty f$.

Goal: To find a way to make sense of improper integrals of above types !!

[Notation: $a,b\in\mathbb{R}$ & always: $a<b$]

Improper integrals of type I:

Let $f\notin\mathcal{R}[a,b]$ & suppose $f\in\mathcal{R}[c,b]$ $\forall\ a<c<b$. If $\lim\limits_{c\to a^+}\int_c^b f$ exists, then we say that the I.I (improper integral) $\int_a^b f$ converges & say

$$\int_a^b f=\lim_{c\to a^+}\int_c^b f.\qquad \left(\because \int_a^b f=\lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f\right). \quad \text{(A)}$$

If the limit DNE, then we say $\int_a^b f$ diverges.

$f\in\mathcal{R}[a+\varepsilon,b]$ $\forall\,\varepsilon>0$

$\varepsilon>0$

$a+\varepsilon=c$

a $\quad$ b

"$\lim\limits_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f$"

$\varepsilon\to 0$ or $c\to a^+$

Def: let $f \notin R[a,b]$ & $f \in R[a,c]$ $\forall$ $a<c<b$.

Then $\int_a^b f$ converges if

$$\lim_{c \to b^-} \int_a^c f$$ exists. We write:

$$\int_a^b f = \lim_{c \to b^-} \int_a^c f.$$

I.I. of type I.

$\forall \varepsilon > 0$

OR, Simply: $$\boxed{\int_a^b f = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f.}$$ —— (B)

Also ~~Finally~~: Let $a<c<b$. & suppose $\int_a^c f$ and/or $\int_c^b f$ are I.I. of type-I. (~~either~~ (A) or (B)).

We write $$\int_a^b f := \int_a^c f + \int_c^b f$$ —— (C)

if both I.I. $\int_a^c f$ & $\int_c^b f$ exist.

Otherwise, $\int_a^b f$ diverges.

Def: If $a<c<b$ & f is unbounded ~~discontinuous~~ at $x=c$, then we write

$$\int_a^b f := \int_a^c f + \int_c^b f$$ provided both

I.I. in the RHS exist.

(B) Kind   (A) Kind.

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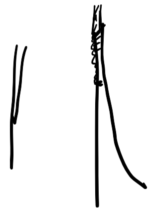
eg:

① $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$.

Clearly, this is an I.I. of type-I. ← $\because \frac{1}{x^2} \notin \mathcal{R}[0,1]$.

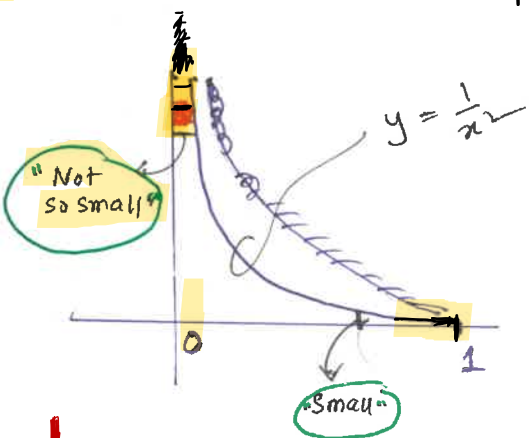
Now, $\displaystyle\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x^2}\,dx$ ← $\because \frac{1}{x^2} \in \mathcal{R}[\varepsilon, 1]$ $\forall\, \varepsilon > 0$ small.

$$= \lim_{\varepsilon \to 0^+} \left[-\frac{1}{x}\right]_\varepsilon^1$$ ← Why? FTC?



$$= \lim_{\varepsilon \to 0^+} \left(\frac{1}{\varepsilon} - 1\right) = +\infty.$$ ← why?

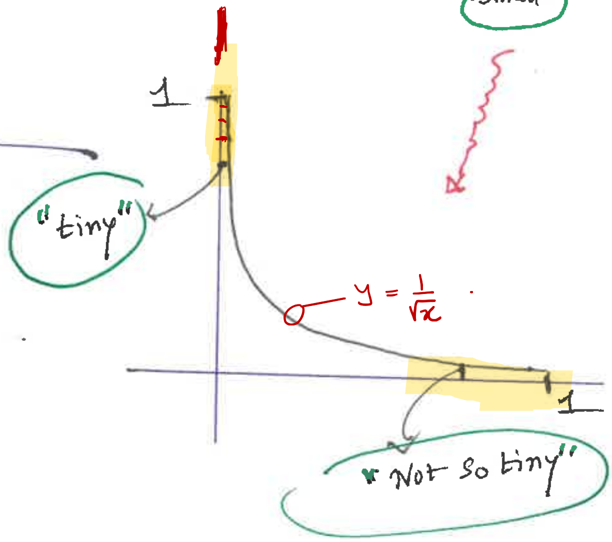
$\therefore \displaystyle\int_0^1 \frac{1}{x^2}\,dx$ diverges.



dx

$$\int_a^b f = \int_a^b f(x)\,dx = \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum f\left(a + \frac{k}{n}\right) = \int f(x)\,dx$$

② $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$. ← tells you we have a chance!!



Now $\forall\, \varepsilon > 0$, $\displaystyle\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$

$$= 2 \times \left[x^{1/2}\right]_\varepsilon^1 = 2\left(1 - \sqrt{\varepsilon}\right).$$

$$\therefore \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0^+} 2\left(1 - \sqrt{\varepsilon}\right) = 2.$$

$\therefore \int_0^1 \frac{1}{\sqrt{x}}\,dx$ is Convergent &

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

(3) $\int_0^2 \frac{1}{2x-x^2}\,dx$.

not defined

$x \xmapsto{f} \frac{1}{2x-x^2}$ is unbdd at $x = 0, 2$.

$2-\varepsilon \rightarrow$

1 2

$\therefore$ We need to investigate

$$\int_0^1 f + \int_1^2 f.$$

or any $0 < c < 2$.

Now $\int_1^2 f = \lim_{\varepsilon \to 0^+} \int_1^{2-\varepsilon} \frac{1}{x(2-x)}\,dx.$

$\frac{1}{x(2-x)} = \left(\frac{1}{x} + \frac{1}{2-x}\right)\frac{1}{2}$

$$= \frac{1}{2} \lim_{\varepsilon \to 0^+} \int_1^{2-\varepsilon} \left(\frac{1}{x} + \frac{1}{2-x}\right)dx.$$

$$= \frac{1}{2} \lim_{\varepsilon \to 0^+} \left[\ln\left(\frac{x}{2-x}\right)\right]_1^{2-\varepsilon}$$

$$= \frac{1}{2} \lim_{\varepsilon \to 0^+} \ln\left(\frac{2-\varepsilon}{\varepsilon}\right) = \infty$$ ← why?

$\Rightarrow \int_0^2 \frac{1}{2x-x^2}\,dx$ diverges.

(4) P.T. $\int_0^1 \frac{dx}{\sqrt{1-x}} = 2.$ (HW)

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In general:

Suppose $p > 0$.

Then $\displaystyle\int_0^1 \frac{dx}{x^p} = ??$ ← Type I

Note that $\forall\ 0 < \varepsilon < 1$,

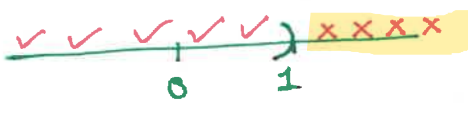
$$\int_\varepsilon^1 \frac{1}{x^p}\,dx = \begin{cases} \left[\dfrac{x^{-p+1}}{-p+1}\right]_\varepsilon^1 & \text{if } p \neq 1 \\ \left[\log x\right]_\varepsilon^1 & \text{if } p = 1. \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-p} \times \left(1 - \varepsilon^{1-p}\right) & \text{if } p \neq 1 \\ -\log \varepsilon & \text{if } p = 1 \end{cases}$$

$$\therefore \lim_{\varepsilon \to 0^+} \int \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p} & 0 < p < 1 \\ \infty & \text{if } p \geq 1. \end{cases}$$

$\therefore$ If $p > 0$, then the I.I. $\displaystyle\int_0^1 \frac{dx}{x^p}$ Converges for $0 < p < 1$ & diverges for $p \geq 1$.

[ZZ: $p < 0$ Case is not worth it !! — why?].



~~Thm (Comparison test – II)~~

~~Result~~:

Thm: Let $\int_a^b f$ be an I.I. at b. Natural classification. Then $\int_a^b f$ converges $\iff$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\left| \int_c^d f \right| < \varepsilon \quad \forall\, c \,\&\, d \text{ s.t. } b-\delta < c < d < b.$$

$b-\delta$ c d b

δ-band.

Proof:

"$\Rightarrow$" Let $\int_a^b f$ converges.

Set $F(x) := \int_a^x f(t)\,dt$. $x \in [a, b)$.

Recall: $f \in \mathcal{R}[a, b-\varepsilon]$ $\forall\, \varepsilon > 0$. ~~small~~.

By assumption: $\lim_{x \to b^-} F(x)$ exists.

$\therefore$ For $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\underbrace{|F(c) - F(d)|}_{\int_c^d f} < \varepsilon \quad \forall\, c, d \text{ s.t. } b-\delta < c < d < b. \quad (*)$$

$$\Rightarrow \left| \int_c^d f \right| < \varepsilon \qquad \text{— '' —}.$$

"$\Leftarrow$" $(*)$ holds $\Rightarrow$?? – HW $\lim_{x \to b^-} F(x)$ exists.

$$\Rightarrow \lim_{x \to b^-} \int_a^b f \text{ exists.}$$

$\boxed{\text{Q.E.D.}}$

Thm: (Comparison test – I)

Let $0 \le f(x) \le g(x) \quad \forall \; x \in [a,b)$. Assume that

$\int_a^b f$ & $\int_a^b g$ are I.I. at b.

① If $\int_a^b g$ converges, then $\int_a^b f$ converges.

② If $\int_a^b f$ diverges, then $\int_a^b g$ diverges.

Proof: $\because$ ① $\Rightarrow$ ②, we only prove ①.

Set $F(x) := \int_a^x g. \quad \forall \; x \in [a,b)$.

$\because \; g \geq 0, \quad F \uparrow$ on $[a,b)$.

$\because \int_a^b g$ converges, we have:

$$\int_a^b g = \lim_{x \to b^-} F(x)$$

$$= \sup \{ \underbrace{F(x)}_{\int_a^x g(x)} : x \in [a,b) \}$$

$$= \sup \left\{ \int_a^x g(x) : x \in [a,b) \right\}.$$

Now $0 \leq f(x) \leq g(x) \quad \forall \; x \in [a,b)$

$$\Rightarrow \; 0 \leq \int_a^x f \leq \int_a^x g. \qquad \forall \; x \in [a,b).$$

$$\leq \int_a^b g.$$

$\therefore \quad x \longmapsto \int_a^x f \quad \uparrow$ on $[a,b)$

& $\quad 0 \leq \int_a^x f \leq \int_a^x g \leq \int_a^b g.$

Then $\quad 0 \leq \int_a^x f \leq \int_a^b g \quad$ & $\quad x \longmapsto \int_a^x f \uparrow$

$\Rightarrow \quad \lim_{x \to b^-} \int_a^x f$ exists & $\int_a^b f \leq \int_a^b g.$

$\boxed{///}$

eg: For $p > 0$, Consider $\int_0^{\pi/2} \frac{\sin x}{x^p} \, dx.$

We know $\quad x \longmapsto \frac{\sin x}{x}$ is bdd &

$$\frac{\sin x}{x} \leq 1 \qquad \forall \; 0 < x \leq \pi/2$$

$$\Rightarrow \quad \frac{\sin x}{x^p} \leq \frac{1}{x^{p-1}} \qquad \text{---}''\text{---}$$

Now $\int_0^{\pi/2} \frac{1}{x^{p-1}} \, dx$ Converges only when $p - 1 < 1$

$\Leftrightarrow \quad p < 2.$

$\therefore$ By Comparison test:

$\int_0^{\pi/2} \frac{\sin x}{x^p} \, dx$ Converges for $p < 2$

& diverges for $p \geq 2.$

$\boxtimes$