examples of Type-II Comparison test :

eg: ① $\displaystyle\int_a^\infty \frac{dx}{e^x+1}$.

Note that $0 < \underbrace{\frac{1}{e^x+1}}_{f(x)} \le \underbrace{\frac{1}{e^x}}_{g(x)}$ $\quad \forall\, x \in \mathbb{R}$.

Now $\int_0^\infty 1/e^x$, & hence $\displaystyle\int_a^\infty \frac{1}{e^x}$ Converges $\forall\, a \in \mathbb{R}$.

~~other page~~

$$\left[\because \lim_{R\to\infty}\int_0^R \frac{1}{e^x}\,dx = \lim_{R\to\infty}\int_0^R e^{-x}\,dx = \lim_{R\to\infty}\left(1-e^{-R}\right) = 1\right.$$

$$\left.\Rightarrow \int_0^\infty \frac{1}{e^x}\,dx = 1\right].$$

$$\Rightarrow \int_a^\infty \frac{dx}{e^x+1} \quad \text{Converges} \quad (\text{by the Comparison test}).$$

② $\displaystyle\int_0^\infty e^{-x^2}\,dx$. $\quad\leftarrow$ Asking for Convergency.

[Euler-Poisson integral & Value $= \sqrt{\pi}/2$].

*

Note that $f(x) := e^{-x^2}$ is in $\mathcal{R}[0,R]$, $\forall\, R>0$. $\hookrightarrow \mathcal{R}[0,\infty)$

Now $e^{x^2} > x^2 \quad \forall\, x \in \mathbb{R}$. $\quad\leftarrow$ Why?

$$\Rightarrow 0 < \frac{1}{e^{x^2}} < \frac{1}{x^2} \quad \forall\, x \neq 0.$$

Now $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ Converges. $\quad\leftarrow (p = 2 > 1$ Case)

$\therefore$ by Comparison test, $\displaystyle\int_1^\infty \frac{1}{e^{x^2}}\,dx$ converges.

$$\Rightarrow \int_0^\infty \frac{1}{e^{x^2}}\,dx \quad \text{converges} \quad [\because e^{-x^2} \in C[0,1]].$$

$\boxtimes$

Thm: (Limit Comparison test – II):

Suppose $f, g \in \mathcal{R}[a,\infty)$ & $f(x), g(x) \geq 0 \ \forall\, x \in [a,\infty)$.

If $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell > 0$, then $\int_a^\infty f$ & $\int_a^\infty g$ converge or diverge together.

(proof is similar to Type-I Case).

Proof: Fix $\varepsilon > 0$ s.t. $\ell - \varepsilon > 0$. $[\because \ell > 0]$

$\because \displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell$, $\exists\ M > 0$ s.t.

$$\left| \frac{f(x)}{g(x)} - \ell \right| < \varepsilon \quad \forall\ x > M.$$

$$\Rightarrow (\ell-\varepsilon) \leq \frac{f(x)}{g(x)} < \ell+\varepsilon \quad \forall\ x > M$$

$$\Rightarrow (\ell-\varepsilon)\, g(x) < f(x) < (\ell+\varepsilon)\, g(x) \quad \forall\ x > M.$$

Suppose $\int_a^\infty f$ Converges. Since $(\ell-\varepsilon)\, g(x) \geq 0 \ \forall\ x > M$

$\underset{\text{Comparison test}}{\Longrightarrow} \int_a^\infty (\ell-\varepsilon)\, g$ Conv. $\Rightarrow \int_a^\infty g$ Converges.

If $\int_a^\infty f$ diverges, then $f(x) < (\ell+\varepsilon)\, g(x)$

$\Rightarrow \dfrac{1}{\ell+\varepsilon} f(x) < g(x) \quad \forall\ x > M$

$\underset{\text{Comparison test}}{\Longrightarrow} \int_a^\infty g$ diverges.

~~$\therefore \int_a^\infty f$ Conv. $\Leftrightarrow \int_a^\infty g$ Conv.~~ The rest is HW.

Hly divergence part.

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eg: $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$.

Let $f(x) = \dfrac{1}{x\sqrt{x^2+1}}$ & $g(x) = \dfrac{1}{x^2}$.

Note that $f(x) = \dfrac{1/x^2}{\sqrt{1+\frac{1}{x^2}}}$

$\therefore$ $f(x), g(x) \geq 0 \quad \forall\ x \in [1,\infty)$.

Note: $f, g \in R[1,\infty)$

Now $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

$\displaystyle = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$

$= 1 > 0.$

Now $\displaystyle\int_1^\infty g = \int_1^\infty \frac{1}{x^2}\,dx$ Converges as $p = 2 > 1$.

$\therefore$ By Comparison test, $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges. ☒

Thm: Let $f \in R[a,\infty)$. If $\displaystyle\int_a^\infty |f|$ converges (i.e. $\displaystyle\int_a^\infty f$ is absolutely convergent), then $\displaystyle\int_a^\infty f$ converges.

AC ⇒ converg.

Proof: (Very similar to $\displaystyle\int_a^b f$ case, i.e., type I case):

Q: Converg. $\overset{?}{\Rightarrow}$ AC. ?

$-|f(x)| \leq f(x) \leq |f(x)| \quad \forall\ x \in [a,\infty)$

$\Rightarrow 0 \leq f(x) + |f(x)| \leq 2|f(x)|$ —(1)—

$\because$ $\displaystyle\int_a^\infty |f|$ Converges, by Comparison test, $\displaystyle\int_a^\infty (f+|f|)$ Converges. Hence $\displaystyle\int_a^\infty f \left(= \int_a^\infty (f+|f|) - \int_a^\infty |f|\right)$ Converges. ☒

Some useful integral tests: We assume $f \in \mathcal{R}[a,\infty)$.

Thm: (Cauchy's test): $\int_a^\infty f$ Converges $\Leftrightarrow$ for $\varepsilon > 0$ $\exists$ $M_0 > 0$

s.t. $\left| \int_{R_1}^{R_2} f \right| < \varepsilon \quad \forall \; R_1, R_2 > M_0$.

[Recall ① [Cauchy's limit criterion]: $\lim_{x \to a} f(x)$ exists $\Leftrightarrow$ for $\varepsilon > 0$

$\exists$ $\delta > 0$ s.t. $|f(x_1) - f(x_2)| < \varepsilon \quad \forall \; x_1, x_2 \in (a-\delta, a+\delta) \setminus \{a\}$.]

~~Proof: $\forall$ $x \geq a$, define $F(x) := \int_a^x f(t)\,dt$.~~

② Def: We say $\lim_{x \to \infty} f(x) = \ell \in \mathbb{R}$ if for $\varepsilon > 0$ $\exists$ $M_0 > 0$

s.t. $|f(x) - \ell| < \varepsilon \quad \forall \; x > M_0$.

③ (Cauchy's Criterion): $\lim_{x \to \infty} f(x) = \ell$ exists $\Leftrightarrow$ for $\varepsilon > 0$

$\exists$ $M_0 > 0$ $\ni$. $|f(x_1) - f(x_2)| < \varepsilon \quad \forall \; x_1, x_2 > M_0$.

— HW —.

Proof: Note that $\int_a^\infty f = \lim_{R \to \infty} \int_a^R f(t)\,dt$. — ⊛

By defn. (& if exists).

$\therefore$ ⊛ exists $\Leftrightarrow$ for $\varepsilon > 0$ $\exists$ $M_0 > 0$ s.t.

The Cauchy Criterion.

$$\underbrace{\left| \int_a^{R_1} f(t)\,dt - \int_a^{R_2} f(t)\,dt \right|}_{= \left| \int_{R_1}^{R_2} f \right|} < \varepsilon \quad \forall \; R_1, R_2 > M_0$$

$$\Leftrightarrow \left| \int_{R_1}^{R_2} f \right| < \varepsilon \quad \forall \; R_1, R_2 > M_0.$$

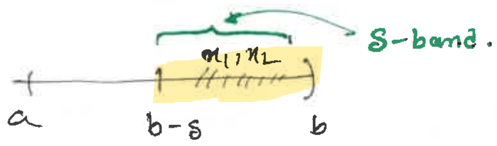
▨

Deviation: on Page 68

Cauchy's Criterion: Suppose $f : (a, b) \longrightarrow \mathbb{R}$ be a fn.. Then

$\lim_{x \to b^-} f(x)$ exists $\Longleftrightarrow$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $a < b - \delta$ &

$$|f(x_1) - f(x_2)| < \varepsilon \quad \forall \; x_1, x_2 \text{ s.t. } b - \delta < x_1 < x_2 < b$$



— HW (Similar proof).

Thm: Let $\int_a^b f$ be an I.I. at b. Then $\int_a^b f$ Converges

$\Longleftrightarrow$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $a < b - \delta$ &

$$\left| \int_{x_1}^{x_2} f \right| < \varepsilon \quad \forall \; b - \delta < x_1 < x_2 < b.$$

Proof: We know $\int_a^b f = \lim_{x \to b^-} \int_a^x f(t)\, dt.$ (if exists).

$\therefore$ $\int_a^b f$ exists $\Longleftrightarrow$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $a < b - \delta$

$$\& \quad \left| \int_a^{x_1} f - \int_a^{x_2} f \right| < \varepsilon \quad \forall \; b - \delta < x_1 < x_2 < b.$$

$$\Longleftrightarrow \quad \left| \int_{x_1}^{x_2} f \right| < \varepsilon \qquad \text{—— " ——}.$$

///

$\therefore$ The above is the Cauchy criterion for I.I. of Type − I.

Back to Type II

Thm: (A.C. test):

Suppose $\varphi \in \mathcal{B}[a,\infty)$ ~~$\cap \mathcal{R}[a,\infty)$~~. If $\int_a^\infty f$ is A.C. then $\int_a^\infty \varphi f$ is also A.C.

[∴ "Scaling" by a bdd fn. is okay for A.C.]

Proof: Note that

$$|(\varphi f)(x)| = |\varphi(x)|\,|f(x)|$$
$$\leq \underbrace{\left(\sup_{x\in[a,\infty)} |\varphi(x)|\right)}_{:=M} \times |f(x)| \quad \forall x \in [a,\infty)$$

~~$0 \leq f$~~
$f, \varphi \geq 0$
$f \in \mathcal{R}[a,b]$, $\varphi \in \mathcal{B}[a,b]$
$\Rightarrow \varphi f \in \mathcal{R}[a,b]$

$$\Rightarrow 0 \leq |\varphi(x) f(x)| \leq M\,|f(x)| \quad \forall x \in [a,\infty).$$

$\Longrightarrow \varphi f \in \mathcal{R}[a,\infty)$ as $f \in \mathcal{R}[a,\infty)$.

'$\varphi f \in \mathcal{R}[a,b]$ $\forall b > a$.'

Finally, as $\int_a^\infty |f|$ converges, by comparison test $\int_a^\infty |\varphi|\,|f|$ also converges. ☐

Now we discuss two important integral tests: ← Non-AC.

Thm: (Abel's test):

Let $\varphi \in \mathcal{B}[a,\infty)$ & suppose φ is monotonic. If $\int_a^\infty f$ converges, then $\int_a^\infty \varphi f$ also converges.

Scaling by bdd monotonic fn. is OK!

Proof: WAIT.

2nd: Dirichlet test.

BUT: We need to prepare the necessary ~~groundwork~~ background!!

~~Back to~~ DEVIATION. Pause for Riemann integration.

Actually, we need the 2^{nd} MVT. But, we can't avoid 1^{st} !!

Thm: (First MVT for integrals):

Let $f, g \in \mathcal{R}[a,b]$ & let f keeps the same sign over $[a,b]$.

Then $\exists$ $\zeta \in [\inf g, \sup g]$ s.t.

$$\int_a^b fg = \zeta \int_a^b f.$$

Curious equality indeed!!

(Also known as weighted MVT!!)

Proof: WLOG, assume that $f(x) > 0$ $\forall$ $x \in [a,b]$.

[OR, consider $-f$].

We know $m \leq g(x) \leq M$

Here $m = \inf g$, $M = \sup g$ over $[a,b]$.

$\because$ $f > 0$, we have:

$$m f(x) \leq g(x) f(x) \leq M f(x) \quad \forall x.$$

$\because$ $f, g, fg \in \mathcal{R}[a,b]$, it follows that

$$m \int_a^b f \leq \int_a^b fg \leq M \int_a^b f.$$

$\Rightarrow$ $\exists$ $\zeta \in [m, M]$ s.t. $\displaystyle\int_a^b fg = \zeta \int_a^b f.$ ▨

\# If $g \in C[a,b]$, then $\zeta = g(c)$ for some $c \in [a,b]$.

$$\therefore \int_a^b fg = g(c) \int_a^b f.$$

Very useful equality!!

\# If $f \equiv 1$ on $[a,b]$, then $\displaystyle\int_a^b g = g(c)(b-a)$.

$g \in C[a,b]$ &

i.e. $g(c) = \dfrac{1}{b-a} \displaystyle\int_a^b g$. MVT !! [We know this.]

$\frac{1}{x} = t$ $\quad \int \frac{-t^2}{\sqrt{(t^2-1)(t^2-r)}} \times \frac{1}{t^2} dt$

eg: Let $r \in (0,1)$. Then

standard application.

$$\frac{\pi}{6} \leqslant \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-rx^2)}} \leqslant \frac{\pi}{6} \frac{1}{\sqrt{1-r/4}}.$$

difficult to compute. So we estimate: practical approach!!

Set $\left.\begin{aligned} f(x) &= \frac{1}{\sqrt{1-x^2}} \\ g(x) &= \frac{1}{\sqrt{1-rx^2}} \end{aligned}\right\}$ $x \in [0, 1/2]$.

Clearly, $f, g \in C[0,1]$. Also $f(x) > 0 \quad \forall x \in [0, 1/2]$.

$\therefore$ By 1^{st} MVT, $\exists$ $\xi \in [0, 1/2]$ ~~$\xi \in [\inf g, \sup g]$~~ s.t.

$$\int_0^{1/2} fg = g(\xi) \int_0^{1/2} f.$$

$$= \frac{1}{\sqrt{1-r\xi^2}} \underbrace{\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx}_{= \pi/6}. \quad (*)$$

(HW)

$\because$ $\xi \in [0, 1/2]$, $\xi^2 \leqslant \frac{1}{4} \Rightarrow r\xi^2 \leqslant \frac{r}{4}$. $(\because r \in (0,1))$

$$\Rightarrow \frac{1}{\sqrt{1-r\xi^2}} \leqslant \frac{1}{\sqrt{1-r/4}}.$$

$$\therefore \int_0^{1/2} fg \leqslant \frac{\pi}{6} \times \frac{1}{\sqrt{1-r/4}}. \quad [\text{by } (*)]$$

Finally, since $r \in (0,1)$, $\frac{1}{\sqrt{1-x^2}} \leqslant (fg)(x) \Rightarrow \frac{\pi}{6} \leqslant \int_0^{1/2} fg$.

$$\therefore \frac{\pi}{6} \leqslant \int_0^{1/2} fg \leqslant \frac{\pi}{6} \frac{1}{\sqrt{1-r/4}}.$$