Jaydeb Sarkar

Proposition: Let $f \in \mathcal{B}[a,b]$ & $P, \tilde{P} \in \mathcal{P}[a,b]$. If $\tilde{P} \supset P$, then

$$\underbrace{L(f,P) \leq L(f,\tilde{P})}_{①} \overset{\text{Known.}}{\leq} \underbrace{U(f,\tilde{P}) \leq U(f,P)}_{②}.$$

getting more ~~closer, if~~ closer!!

Proof: "$L(f,\tilde{P}) \leq U(f,\tilde{P})$" is known.

$\therefore$ Enough to prove "$L(f,P) \leq L(f,\tilde{P})$" &

"$U(f,\tilde{P}) \leq U(f,P)$".

We only prove the 1^{st} one (as the 2^{nd} one will be similar).

First, assume that $\tilde{P} := P \cup \{\tilde{x}\}$,

~~Where~~ where $\tilde{x} \in [a,b] \setminus P$. [$\because$ $\tilde{x}$ a new node.]

Set $P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

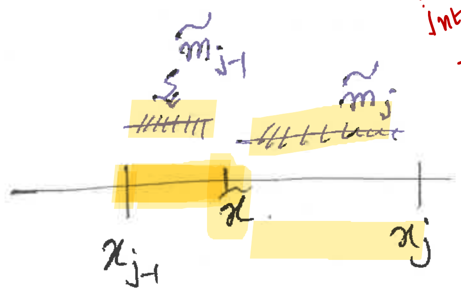
Then $\exists$ $j \in \{1, \ldots, n\}$ s.t.

$$x_{j-1} < \tilde{x} < x_j.$$

$\uparrow$ Int I_j

[$\because$ $x \in [a,b] \setminus \{a,b\}$]

$\tilde{x} \in (a,b)$ & $\tilde{x} \neq x_j$ $\forall j = 1, \ldots, n$



$\tilde{P} : a = x_0 < x_1 < \cdots < x_{j-1} < \tilde{x} < x_j < \cdots < x_n = b$

Set: $\tilde{m}_{j-1} := \inf\{f(x) : x \in [x_{j-1}, \tilde{x}]\}$

& $\tilde{m}_j := \inf\{f(x) : x \in [\tilde{x}, x_j]\}$.

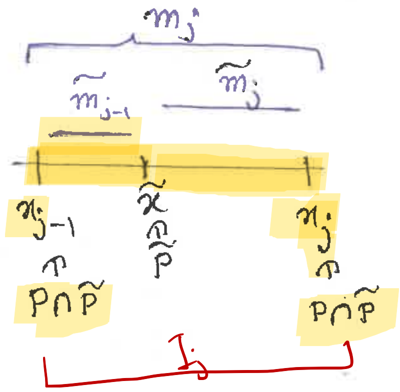
$m_1 \times |I_1| + \cdots + m_{j-1}|I_{j-1}| + m_j |I_j| + m_{j+1}|I_{j+1}| + \cdots + m_n |I_n|$

$$\therefore\ L(f,\tilde{P}) - L(f,P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j (x_j - \tilde{x}) - m_j (x_j - x_{j-1}).$$

$I_j \supseteq [x_{j-1}, \tilde{x}]$

$m_j = \inf_{I_j} f(x) \leq \inf_{[x_{j-1},\tilde{x}]} f = \tilde{m}_{j-1}$

Now



$\Longrightarrow\ m_j \leq \tilde{m}_{j-1},\ \tilde{m}_j$

$\because$ inf over ~~sub~~ "smaller" subsets.

$$\Rightarrow L(f,\tilde{P}) - L(f,P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j (x_j - \tilde{x}) - m_j (\tilde{x} - x_{j-1}) - m_j (x_j - \tilde{x}).$$

$$[\because\ x_j - x_{j-1} = (x_j - \tilde{x}) + (\tilde{x} - x_{j-1})]$$

$$= \underbrace{(\tilde{m}_{j-1} - m_j)}_{\geq 0}(\tilde{x} - x_{j-1}) + \underbrace{(\tilde{m}_j - m_j)}_{\geq 0}(x_j - \tilde{x})$$

$$\geq 0$$

$$\Rightarrow L(f,\tilde{P}) \geq L(f,P).$$

The general case: by induction.

The upper sum case: Similar & HW.

$\boxed{//}$

Cor: Let $f \in \mathcal{B}[a,b]$ & $P, Q \in \mathcal{P}[a,b]$. Then

$$L(f,P) \leq U(f,Q).$$

Proof: Let $\tilde{P} := P \cup Q \Rightarrow \tilde{P} \supset P, Q$.

$$\therefore\ L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,Q)$$

By applying the above prop. for $\tilde{P}, P$ & $\tilde{P}, Q$. where $X = P$ & Q.

In particular: $L(f,P) \leq U(f,Q)$. $\boxed{//}$

$L(f,P) \leq U(f,\tilde{P})$
$U(f,\tilde{P}) \leq U(f,Q)$

Jaydeb Sarkar.

Cor: If $f \in \mathcal{B}[a,b]$, then

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f .$$

Proof: We know: $L(f, P_1) \leqslant U(f, P_2) \quad \forall P_1, P_2 \in \mathcal{P}[a,b]$.

$\therefore$ For a fixed $P_2 \in \mathcal{P}[a,b]$,

$$\underline{\int_a^b} f = \sup_{P_1 \in \mathcal{P}[a,b]} L(f, P_1) \leqslant U(f, P_2).$$

$\therefore$ Taking inf on all over $P_2 \Rightarrow \underline{\int} f \leqslant \inf_{P_2} U(f, P_2) = \overline{\int} f.$

$\boxtimes$

Notation: $\mathcal{R}[a,b] = \{ f \in \mathcal{B}[a,b] : f \text{ is Riemann integrable.} \}$.

Fact: Suppose $f \in \mathcal{B}[a,b]$. Then

$$f \in \mathcal{R}[a,b] \Longleftrightarrow \underline{\int_a^b} f \geqslant \overline{\int_a^b} f .$$

$f \notin \mathcal{R}[a,b]$

$\Leftrightarrow \underline{\int} f \neq \overline{\int} f$

$\Leftrightarrow \underline{\int} f < \overline{\int} f$

Q: $\mathcal{B}[a,b] = \mathcal{R}[a,b]$?

Ans: No!

eg: Consider the Dirichlet fn.: $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1] . \end{cases}$$

Clearly, $f \in \mathcal{B}[0,1]$.

Suppose $P : 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$.

Recall: $I_j := [x_{j-1}, x_j]$.

$\Rightarrow$ $I_j \cap \mathbb{Q} \neq \varphi$ & $I_j \cap \mathbb{Q}^c \neq \varphi$. $\forall j = 1, \dots, n$.

$\Rightarrow$ $m_j = 0$ & $M_j = 1$ $\forall j = 1, \dots, n$.

$\therefore$ $L(f, P) = 0$ & $U(f, P) = 1$. [By the defn's. of L & U].

$L(f,P) = \sum \underbrace{m_j}_{=0} |I_j| = 0$.

$U(f,P) = \sum \underbrace{M_j}_{=1} |I_j| = \sum |I_j| = 1 - 0 = 1$.

$\forall P \in \mathcal{P}[0,1]$.

$\Rightarrow$ $\underline{\int_0^1} f = 0 \neq 1 = \overline{\int_0^1} f$.

$f \in R[0,1]$ if $\underline{\int} f = \overline{\int} f$. $(\geq)$

$\underline{\int} f = \sup_{P \in \mathcal{P}[0,1]} L(f,P) = 0$ [$\because L(f,P) = 0 \ \forall P$.]

$\Rightarrow$ $f \notin \mathcal{R}[0,1]$. $\boxtimes$

$\therefore \mathcal{B}[a,b] \supsetneq \mathcal{R}[a,b]$.

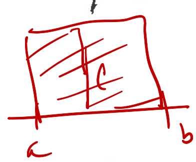
eg: ($\mathcal{R}[a,b] \neq \varphi$):

Fix $c \in \mathbb{R}$ & define $f(x) = c$ $\forall x \in [a,b]$.

If $P \in \mathcal{P}[a,b]$, $M_j = c$ $\forall j \Rightarrow$ & $m_j = c$

Then, $\forall P \in \mathcal{P}[a,b]$, $L(f,P) = c \times (b-a) = U(f,P)$.

$\uparrow$ Why? check.



$\Rightarrow$ $\underline{\int_a^b} f = c \times (b-a) = \overline{\int_a^b} f$

$\Rightarrow$ $f \in \mathcal{R}[a,b]$ & $\int_a^b f = c(b-a)$. $\boxtimes$

eg: $\exists f$ s.t. $|f| \in \mathcal{R}[a,b]$ but $f \notin \mathcal{R}[a,b]$.

Consider $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$.

$\underline{\int} f = -1 \neq 1 = \overline{\int} f$.

$\Rightarrow f \notin \mathcal{R}[0,1]$.

Clearly, $f \in \mathcal{B}[0,1]$. Here $|f| \equiv 1 \Rightarrow |f| \in \mathcal{R}[0,1]$.

But $f \notin \mathcal{R}[0,1]$, $\longleftarrow$ HW. $\boxtimes$