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Note

In view of the above example (i.e. $f \notin \mathcal{R}[a,b]$ but $|f| \in \mathcal{R}[a,b]$), we have the following question:

$$f \in \mathcal{R}[a,b] \overset{?}{\Longrightarrow} |f| \in \mathcal{R}[a,b] \quad ?$$

— Think about it —

"$\mathcal{R}[a,b]$ is invariant under $|\cdot|$"?

Tool

Thm: Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff$ for $\varepsilon > 0$ $\exists$ $P \in \mathcal{P}[a,b]$ s.t

$$\underbrace{U(f,P) - L(f,P)}_{\geq 0} < \varepsilon .$$

Fixed $f \in \mathcal{B}[a,b]$, define $\eta_f : \mathcal{P}[a,b] \to \mathbb{R}$
$\eta_f(P) := U(f,P) - L(f,P)$

$\eta_f(P) < \varepsilon$ for some $P \in \mathcal{P}[a,b]$

Proof: "$\Leftarrow$" Let $\varepsilon > 0$.

$\therefore$ $\exists$ $P \in \mathcal{P}[a,b]$ s.t. $\quad U(f,P) - L(f,P) < \varepsilon$.

Now $\quad L(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f,P)$.

(true in general)

$$\Rightarrow \overline{\int_a^b} f \leq U(f,P) < \varepsilon + L(f,P).$$

(By assumption)

But $\quad L(f,P) \leq \underline{\int_a^b} f$

$$\therefore \overline{\int_a^b} f < \varepsilon + \underline{\int_a^b} f$$

$$\Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon .$$

We also know, in general, that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

$$\therefore 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon , \quad \forall \varepsilon > 0.$$

$$\Rightarrow \underline{\int} f = \overline{\int} f \Rightarrow f \in \mathcal{R}[a,b].$$

$\inf_P U(f,P) \rightsquigarrow \overline{\int} f = \int f = \int_a^b f$

$\Rightarrow$ $\underline{\int} f \rightsquigarrow \sup_P L(f,P)$

$\sup A = m$
For $\varepsilon > 0$ $\exists x \in A$ s.t. $x > m - \varepsilon$

"$\Rightarrow$" Suppose $f \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$.

$\therefore \exists P_1 \in \mathcal{P}[a,b]$ $\ni$ $L(f,P_1) > \underline{\int_a^b} f - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2}$. — (A)

& $\exists P_2 \in \mathcal{P}[a,b]$ $\ni$ $U(f,P_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}$. — (B)

Set $P := P_1 \cup P_2$. $\Rightarrow$ $P \supset P_1$ & P_2.

common refinement of

Claim: $U(f,P) - L(f,P) < \varepsilon$.

Now, $U(f,P) \leq U(f,P_2) < \int_a^b f + \frac{\varepsilon}{2}$.

$(\because P \supset P_2)$ (B)

$< L(f,P_1) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

(A)

$\leq L(f,P) + \varepsilon$.

$(\because P \supset P_1)$

$\Rightarrow U(f,P) - L(f,P) < \varepsilon$. ◻

Note: We always have the following:

$$0 \leq U(f,P) - L(f,P) \;!!$$

Def: Let $P \in \mathcal{P}[a,b]$. Then the mesh of P (or norm of P) is defined as:

$$\|P\| := \max\{x_j - x_{j-1} : 1 \leq j \leq n\},$$

where $P: a = x_0 < x_1 < \cdots < x_n = b$.

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A kind of "continuity" property of Riemann integ.

Thm: (Darboux thm) Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \quad (\text{for some } P \in \mathcal{P}[a,b])$$

$$\forall\ P \in \mathcal{P}[a,b] \text{ with } \|P\| < \delta.$$

$(\mathcal{P}[a,b], \|\cdot\|)$

$\|\cdot\| : \mathcal{P}[a,b] \to \mathbb{R}_{\geq 0}$

[For $f \in \mathcal{B}[a,b]$ fixed, define $\eta : \mathcal{P}[a,b] \to \mathbb{R}_{\geq 0}$ by

$$\eta(P) = U(f,P) - L(f,P) \quad \forall\ P \in \mathcal{P}[a,b].$$

So, $f \in \mathcal{R}[a,b] \iff$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\eta(P) = |\eta(P)| < \varepsilon \quad \forall\ \|P\| < \delta \quad !! \ ]$$

$F : [a,b] \to \mathbb{R}$ is cont. at c

for $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$|F(x) - F(c)| < \varepsilon$

$\forall\ |x - c| < \delta$

$(\mathbb{R}_{\geq 0}, |\cdot|)$

$\uparrow \eta$

$(\mathcal{P}[a,b], \|\cdot\|)$

Proof: "$\Leftarrow$" : Follows from ~~prev~~ the last observation.

"$\Rightarrow$" Let $f \in \mathcal{R}[a,b]$ & let $\varepsilon > 0$.

~~Goal: $\exists \delta$ & $P \in \mathcal{P}[a,b] \ni U(f,P) - L(f,P) < \varepsilon$.~~

~~[Again, by the prev.~~ observation, ~~we will conclude that~~]

$\therefore$ $\exists$ $\tilde{P} \in \mathcal{P}[a,b]$ s.t. $U(f,\tilde{P}) - L(f,\tilde{P}) < \frac{\varepsilon}{2}$. $\quad (*)$

(By the prev. obs.)

Assume that $\boxed{\#\text{ nodes of } \tilde{P} = p}$.

$\because$ $f \in \mathcal{B}[a,b]$, $\exists$ $M > 0$ s.t.

$$|f(x)| \leq M \quad \forall\ x \in [a,b]$$

Our friendly bound.

Set $\delta := \dfrac{\varepsilon}{8pM}$. $\leftarrow$ Suggested by a back calculation.

Let $P \in \mathcal{P}[a,b]$ & suppose $\underbrace{\|P\|}_{\text{max length of subintervals.}} < \delta$. (fix P)

Set $\hat{P} := P \cup \tilde{P}$ $\longleftarrow$ $\therefore \hat{P} \supset P, \tilde{P}$.

$\Rightarrow$ $\hat{P}$ has atmost p nodes that are not in P. **

Now, ~~if~~ Let $\hat{P} = P \cup \{\tilde{x}\}$ & $\tilde{x} \notin P$. [i.e. $p=1$ case].

As earlier: Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

(p-12)

& assume $x_{j-1} < \tilde{x} < x_j$.

Then: $L(f,\hat{P}) - L(f,P) = (\tilde{m}_{j-1} - m_j)(\tilde{x} - x_{j-1})$
$+ (\tilde{m}_j - m_j)(x_j - \tilde{x})$

[See page - 12]

$\leq 2M\|P\|$.

$|||^{ly}$, if $\hat{P} = P \cup \{\tilde{x}_1, \ldots, \tilde{x}_p\}$, then (by induction),

$$L(f,\hat{P}) - L(f,P) \leq 2Mp\|P\|$$

$$< 2Mp \times \delta = \frac{\varepsilon}{4}.$$

$$\left.\begin{aligned} \therefore\ L(f,\hat{P}) - L(f,P) &< \frac{\varepsilon}{4}. \\ |||^{ly}\quad U(f,P) - U(f,\hat{P}) &< \frac{\varepsilon}{4} \end{aligned}\right\}$$ $\longrightarrow$ (†) (**)

$\therefore$ (†) $\Rightarrow$ $U(f,P) - L(f,P) < \frac{\varepsilon}{2} + \left(U(f,\hat{P}) - L(f,\hat{P})\right)$ (**)

But (*) $\Rightarrow$ $U(f,\tilde{P}) - L(f,\tilde{P}) < \frac{\varepsilon}{2}$ (***). $\because \hat{P} \supset \tilde{P}$, we know

$L(f,\tilde{P}) \leq L(f,\hat{P})$ & $U(f,\tilde{P}) \geq U(f,\hat{P})$.

$\Rightarrow U(f,\hat{P}) - L(f,\hat{P}) \leq U(f,\tilde{P}) - L(f,\tilde{P}) < \varepsilon/2$. (***) $\Rightarrow U(f,\tilde{P}) - L(f,\tilde{P})$

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$\therefore$ (**) $\Rightarrow U(f,P) - L(f,P) < \varepsilon$. $\boxtimes$

Notation: $C[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} \text{ continuous} \}$

$\{\text{polynomials}\} \subseteq C[a,b]$. AND $\{\text{rationals}\} \subseteq C[a,b]$.

roots of f $\notin [a,b]$

$\frac{p}{q}$, $Z(q) \cap [a,b] = \varphi$

Also: $\{ e^x, \sin x, \cos x, \dots \} \subseteq C[a,b]$.

Who are they ??

$\therefore$ This is a large class !!

$\frac{p}{q}$ w/ $Z(q) \cap [a,b] = \varphi$, $\frac{p}{q} \in C[a,b]$

Any relation $\{ e^x, \sin x, \cos x, \dots \} \overset{??}{\longleftrightarrow} \mathbb{R}[x]$??

A pending question.

Thm: $C[a,b] \subseteq R[a,b]$.

Proof: Let $f \in C[a,b]$.

$\Rightarrow f : [a,b] \longrightarrow \mathbb{R}$ is uniformly continuous.

Let $\varepsilon > 0$. By uniform cont. $\exists \ \delta > 0$ s.t.

$$|f(x) - f(y)| < \underbrace{\frac{\varepsilon}{b-a}}_{\text{The new "}\varepsilon\text{"}} \quad \forall x, y \in [a,b] \text{ s.t. } |x-y| < \delta.$$

Let $P \in \mathcal{P}[a,b]$ & assume $\|P\| < \delta$. Fix P.

Set $P : a = x_0 < x_1 < \cdots < x_n = b$.

Now $f|_{[x_{j-1}, x_j]} : [x_{j-1}, x_j] \longrightarrow \mathbb{R}$ is also unif. cont. $\forall j = 1, \dots, n$.

$\Rightarrow f|_{[x_{j-1}, x_j]}$ assumes it max (which is M_j) & min (which is m_j) in $[x_{j-1}, x_j]$ $\forall j = 1, \dots, n$.

$\because \|P\| < \delta$, we know

The max length of subintervals

$$x_j - x_{j-1} < \delta \quad \forall\, j = 1, \dots, n.$$

In particular: $|x - y| < \delta \quad \forall\, x, y \in [x_{j-1}, x_j] \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \forall\, x, y \in I_j$

$$\therefore \quad M_j - m_j < \frac{\varepsilon}{b-a} \quad \forall\, j = 1, \dots, n.$$

$$\therefore \quad U(f, P) - L(f, P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) - \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$

$$= \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1})$$

$$< \frac{\varepsilon}{b-a} \times \sum_{j=1}^{n} (x_j - x_{j-1})$$

$$= \frac{\varepsilon}{b-a} \times b - a$$

$$= \varepsilon.$$

$$\Rightarrow \quad U(f, P) - L(f, P) < \varepsilon.$$

$$\Rightarrow \quad f \in R[a, b]. \qquad \boxed{\checkmark}$$

Q: Still, how to compute $\int_a^b f$ for $f \in C[a, b]$??

— WAIT —