Jaydeb Sarkar.

On "how to compute $\int_a^b f$ ?" , where $f \in \mathcal{R}[a,b]$

$1^{st}$ attempt: (Sequential approach)

**Thm:** Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff \exists$ a seqn. $\{P_n\} \subseteq \mathcal{P}[a,b]$ s.t.

$$\lim_{n\to\infty} \underbrace{\left[ U(f,P_n) - L(f,P_n) \right]}_{:= r_n} = 0.$$

$\{r_n\} \subseteq \mathbb{R}_{\geq 0}$.   $\#P_n \not\sim n$

Moreover, in this case:

$$\int_a^b f = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n).$$

**Proof:** [ Note: $\because$ $U(f,P) - L(f,P) \geq 0$ $\forall P \in \mathcal{P}[a,b]$, $\{U(f,P_n) - L(f,P_n)\} \subseteq \mathbb{R}_{\geq 0}$. ⊛ ]

$\tilde{P} > P$

$U(f,\tilde{P}) - L(f,\tilde{P}) \leq U(f,P) - L(f,P)$ ✗

"$\Rightarrow$" Let $f \in \mathcal{R}[a,b]$. We know: "for $\varepsilon > 0$, $\exists$ $P_\varepsilon \in \mathcal{P}[a,b]$ $\ni$ $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$." ← Call it "Cauchy Criterion"

$\{P_{\frac{1}{n}}\} \sim \{P_n\}$

$\therefore$ For each $n \in \mathbb{N}$ ($\varepsilon = \frac{1}{n}$), $\exists$ $P_n \in \mathcal{P}[a,b]$ s.t.

$$U(f,P_n) - L(f,P_n) < \frac{1}{n}.$$

By ⊛ above: $U(f,P_n) - L(f,P_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

$\{r_n\} \subseteq \mathbb{R}$
$r_n \to 0$
$\iff$ for $\varepsilon > 0$ $\exists$ $N_0 \in \mathbb{N}$ s.t. $|r_n| < \varepsilon$ $\forall n \geq N_0$

"$\Leftarrow$" If $\lim_{n\to\infty} \left[ U(f,P_n) - L(f,P_n) \right] = 0$, then for $\varepsilon > 0$ $\exists$ $N_0 \in \mathbb{N}$ $\ni$ $U(f,P_{N_0}) - L(f,P_{N_0}) < \varepsilon$ ($\forall n \geq N_0$)

$\Longrightarrow$ $f \in \mathcal{R}[a,b]$. ← By Cauchy Criterion.

Finally, if $f \in \mathcal{R}[a,b]$, then for $\{P_n\} \subseteq \mathcal{P}[a,b]$ as above:

We have:

$$0 \le U(f,P_n) - \overline{\int_a^b} f \qquad \longleftarrow \left(\because \overline{\int} f = \inf_P U(f,P)\right)$$

$$= U(f,P_n) - \int_a^b f \qquad \longleftarrow \because f \in \mathcal{R}[a,b]$$

(As $\underline{\int} f = \sup_P L(f,P)$) $\to$ $\le U(f,P_n) - L(f,P_n)$ $\forall n.$ $\longrightarrow 0$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} U(f,P_n) = \overline{\int_a^b} f = \int_a^b f.$$

Finally, $L(f,P_n) = U(f,P_n) - \big(U(f,P_n) - L(f,P_n)\big)$

$$\downarrow$$

$$\int_a^b f \;-\; 0 \qquad \text{as } n\to\infty$$

Standard argument.

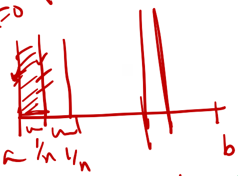
$$\Rightarrow \lim_{n\to\infty} L(f,P_n) = \int_a^b f. \qquad \boxtimes$$

**DANGER!** $a_n - b_n \longrightarrow 0 \not\Longrightarrow$ Even $\lim a_n$ &/or $\lim b_n$ exist!!

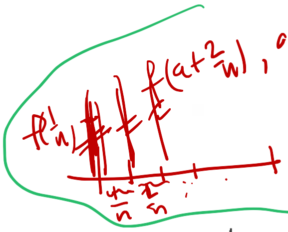
**Remark:** Evidently, if $\exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $L(f,P_n) \longrightarrow c$ & $U(f,P_n) \longrightarrow c$, then $f \in \mathcal{R}[a,b]$ & $\int_a^b f = c$.

$\lim_{n\to\infty} \sum_{j=1}^{n} f(a+\frac{j}{n}) \times \frac{1}{n} = \int_a^b f$

$\lim \left( \sum_{j=0}^{n-1} f(a+\frac{j}{n}) \times \frac{1}{n} \right)$




$\lim_{n\to\infty} \sum_{j=1}^{n} f(a+\frac{j+1}{n})\frac{1}{n}$ " $\int_a^b f$.




A nice way to prove existence of $\mathcal{R}$-integrability (& evaluating too) of bdd fn's. ["Reminding us Newton's"]

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eg: $f(x) = x^2$ $x \in [0,1]$. [BTW: $f \in C[0,1] \Rightarrow f \in \mathcal{R}[0,1]$. But we prove it directly.]

Fix $n \in \mathbb{N}$.

Consider "the classical" partition: Why call it classical?

$P_n: 0 = x_0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = 1.$

$$\therefore I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right] \quad \forall j = 1, \ldots, n.$$

$\because f$ is $\uparrow$, it follows that:

$$m_j = \left(\frac{j-1}{n}\right)^2 \quad \& \quad M_j = \left(\frac{j}{n}\right)^2.$$

$\forall j = 1, \ldots, n.$

$$\therefore U(f, P_n) = \sum_{j=1}^{n} M_j \times \frac{1}{n} = \sum_{j=1}^{n} \frac{j^2}{n^2} \times \frac{1}{n}.$$

$$= \frac{1}{n^3} \sum_{j=1}^{n} j^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) = \frac{1}{n^3} \times \frac{1}{6} \times n \times (n+1)(2n+1)$$

$$= \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Also, $L(f, P_n) = \sum_{j=1}^{n} m_j \frac{1}{n} = \frac{1}{n^3} \sum_{j=1}^{n} (j-1)^2$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right)$$

$$= \frac{1}{n^3} \times \frac{1}{6} \times (n-1) \times n \times \left(2(n-1)+1\right)$$

$$= \frac{1}{6} \times \left(1 - \frac{1}{n}\right) \times \left(2 - \frac{1}{n}\right).$$

$$\therefore U(f, P_n) \longrightarrow \frac{1}{3} \quad \& \quad L(f, P_n) \longrightarrow \frac{1}{3}.$$

$\Rightarrow \quad U(f, P_n) - L(f, P_n) \longrightarrow 0.$

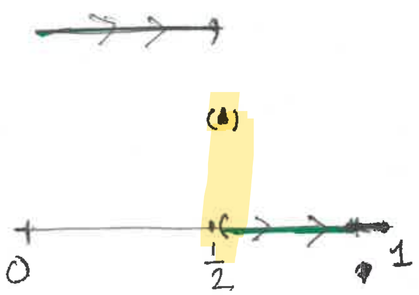
$\Rightarrow \quad f \in \mathcal{R}[0,1]$ & $\int_0^1 f = \lim_{n\to\infty} U(f, P_n) = \frac{1}{3}$. ∎

**Q:** $\mathcal{R}[a,b] \setminus C[a,b] \neq \varphi$ ?

i.e. $\exists \; f \in \mathcal{B}[a,b]$ s.t. $f \in \mathcal{R}[a,b]$ but $f \notin C[a,b]$ ??

**Ans:** Yes.

eg:



$$f(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & \frac{1}{2} < x \le 1. \end{cases}$$

Claim: $f \in \mathcal{R}[0,1]$. Clearly, $f \notin C[0,1]$.

Why? "$\varepsilon$-$\delta$" argument? Please do it

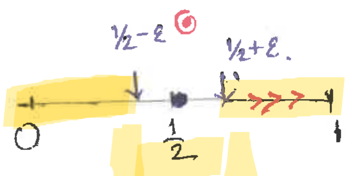
Proof

Let $\varepsilon > 0$ (small).

Consider the partition: $P_\varepsilon : 0 = x_0 < x_1 = \frac{1}{2} - \varepsilon < x_2 = \frac{1}{2} + \varepsilon < x_3 = 1$

i.e: $0 < \frac{1}{2} - \varepsilon < \frac{1}{2} + \varepsilon < 1$.

A 2-node partition.




$\therefore \; I_1 = [0, \frac{1}{2} - \varepsilon]$, $I_2 = [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$, $I_3 = [\frac{1}{2} + \varepsilon, 1]$

$\therefore \; m_1 = 1, \; m_2 = 0, \; m_3 = 0$

$M_1 = 1, \; M_2 = 1, \; M_3 = 0$.

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$$\therefore \ L(f, P_\varepsilon) = \sum_{j=1}^{3} m_j \, |I_j|$$

$$= 1 \times \left(\tfrac{1}{2} - \varepsilon\right) + 0 + 0.$$

$$= \tfrac{1}{2} - \varepsilon.$$

& $$U(f, P_\varepsilon) = \sum_{j=1}^{3} M_j \, |I_j|$$

$$= 1 \times \left(\tfrac{1}{2} - \varepsilon\right) + 1 \times 2\varepsilon + 0.$$

$$= \tfrac{1}{2} + \varepsilon.$$

$$\therefore \ U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon.$$

i.e. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < 3\varepsilon.$

why? $\Rightarrow f \in \mathcal{R}[0,1].$

Q: $\int_0^1 f = ?$

$\therefore$ For $\varepsilon = \frac{1}{2n}$, we have,

$U(f, P_n) - L(f, P_n) = \frac{1}{n} \to 0.$

$\Rightarrow \lim \left[U(f, P_n) - L(f, P_n)\right] = 0.$

$\because U(f, P_n) = \frac{1}{2} + \frac{1}{2n}$

$\Rightarrow \int_0^1 f = \frac{1}{2}.$

Remark: In fact, if $f \in \mathcal{B}[0,1]$ with finitely many discontinuity, then $f \in \mathcal{R}[0,1]$.

— WAIT —

— × —

Let's make it more refined !!

The theory

**Def:** Let $P: a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. ~~Define~~ A fn.

$$T_P : \{I_j\}_{j=1}^{n} \longrightarrow [a,b] \quad \text{is called a } \underline{\text{tag}} \text{ of } P \text{ if}$$

$$T_P(I_j) \in I_j \qquad \forall\, j = 1, \ldots, n.$$

Simply

Given a partition $P: a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, a

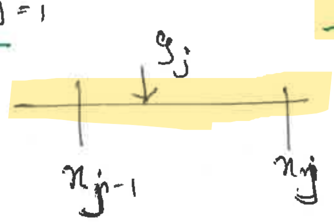
tag or tag set of $P$ is a collection of points $\{g_j\}_{j=1}^{n}$

s.t. $\quad g_j \in I_j \qquad \forall\, j = 1, \ldots, n.$

Even Simpler.

of P / for P / Corresponding to P.

A tag set $T_P = \{g_j\}_{j=1}^{n}$ where $g_j \in I_j \quad \forall\, j = 1, \ldots, n.$




e.g.


$$P: 0 < \tfrac{1}{2} < 1 < 2 < 2\tfrac{1}{2} < 4$$

$$T_P = \{\, 1/6,\ 5/8,\ 4/3,\ 2\tfrac{1}{5},\ 3\tfrac{1}{3} \,\}$$

a tag set.

Note: If $P$ has $n$-nodes [nodes excludes end points $\Rightarrow$ $n+1$) subintervals], then $\#T_P = n+1 \quad \forall$ tag set $T_P$.

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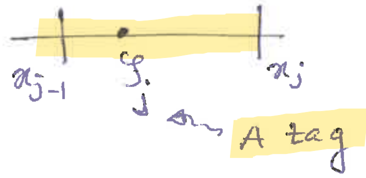
**Def:** Let $f \in B[a,b]$, $P \in \mathcal{P}[a,b]$. & suppose $\boxed{T_P \text{ a tag set of } P.}$

The Riemann sum of $f$ w.r.t. $(P, T_P)$ is defined by:

$$S(f, P) := \sum_{j=1}^{n} f(\xi_j)\, |I_j| \,, \qquad \text{—} \quad \circledast$$

where: $T_P = \{\xi_j\}_{j=1}^{n}$.





Note: ① $S(f, P)$ depends on $T_P = \{\xi_j\}_{j=1}^{n}$.

$T_P$

② LHS of $\circledast$ doesn't involve $T_P$ but it is there !!

③ $\exists$ infinitely many tag sets for a given partition $P$.

④ In fact a question; What is the meaning of Riemann Sum ?

**Fact:** (Answering ④)

Suppose $P: a = x_0 < \cdots < x_n = b$ be a partition &

$T_P = \{\xi_j\}_{j=1}^{n}$ be a tag of $P$.

$$\therefore \quad m_j \le f(\xi_j) \le M_j \qquad \forall\, j = 1, \dots, n$$

$$\Rightarrow \quad m_j\, |I_j| \le f(\xi_j)\, |I_j| \le M_j\, |I_j| \qquad \forall\, j = 1, \dots, n.$$

$$\Rightarrow \quad \sum \le \sum \le \sum$$

$$\Rightarrow \quad L(f, P) \le S(f, P) \le U(f, P).$$

$T_P$ ✗ $\qquad T_P$ ✓ $\qquad T_P$ ✗

$\therefore$ it is a better approximation.

HOWEVER, $S(f, P)$ depends on $T_P$ but $L(f, P)$ & $U(f, P)$ does not !!