$[x_n \to x : \ |x_n - x| < \varepsilon \ \ \forall n \geq N]$

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Def: For $S \subseteq \mathbb{R}$, $\mathcal{B}(S) := \{ f : S \to \mathbb{R} \text{ bdd.} \}$

Set of all bdd fn's on S.

A vector space over $\mathbb{R}$.

[Here $(\alpha f + g)(x) = \alpha f(x) + g(x)$, $\forall \alpha \in \mathbb{R}$, $f, g \in \mathcal{B}(S)$]

Def: $\forall f \in \mathcal{B}(S)$, define $\|f\|$ (read: norm of f)

by $\|f\| = \sup_{x \in S} |f(x)|$.

Also known as "the sup norm".

$S = [0,1]$

$f(x) = x$

$g(x) = 1 - x$

$\|f\| = 1 = \|g\|$

$\|fg\| = \frac{1}{4}$.

Remark:

(1) $\|\cdot\| : \mathcal{B}(S) \to \mathbb{R}_{\geq 0}$.

(2) $\|f\| = 0 \iff f \equiv 0$.

(3) $\|f + g\| \leq \|f\| + \|g\|$ — Δ-inequality

(4) $\|\alpha f\| = |\alpha| \, \|f\| \quad \forall \alpha \in \mathbb{R}$.

(5) $\|fg\| \leq \|f\| \, \|g\|$ — Submultiplicative.

$\forall f, g \in \mathcal{B}(S)$.

Looks like $|\cdot|$ on $\mathbb{R}$!!

$\sup_S \left(|f(x)| \, |g(x)| \right) = \sup_S |f| \times \sup_S |g|$.

Remark: Indeed, $\|\cdot\|$ on $\mathcal{B}(S)$ plays the role of $|\cdot|$ on $\mathbb{R}$!!

Def: $\forall f, g \in \mathcal{B}(S)$, define the distance between f & g

as:

$$d(f, g) = \|f - g\|.$$

Also known as metric on $\mathcal{B}(S)$.

Remark: $\therefore \ d : \mathcal{B}(S) \times \mathcal{B}(S) \to \mathbb{R}_{\geq 0}$. And:

(1) $d(f, g) = \sup_{x \in S} |f(x) - g(x)|$.

(2) $d(f, g) \leq d(f, h) + d(h, g)$.

(3) $d(f, g) = 0 \iff f = g$.

$\forall f, g \in \mathcal{B}(S)$.

Let $\{f_n\} \subseteq \mathcal{F}(S)$ & $f \in \mathcal{F}(S)$. Recall: $f_n \xrightarrow{u} f$ on S if

$\mathcal{F}(S) = \{g : S \to \mathbb{R}\}$

for $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall \; x \in S, \; n \geq N.$$

$$\Updownarrow \quad (\because \; f_n - f \in \mathcal{B}(S) \;\; \forall \, n \geq N)$$

$$\|f_n - f\| < \varepsilon \quad \forall \; n \geq N.$$

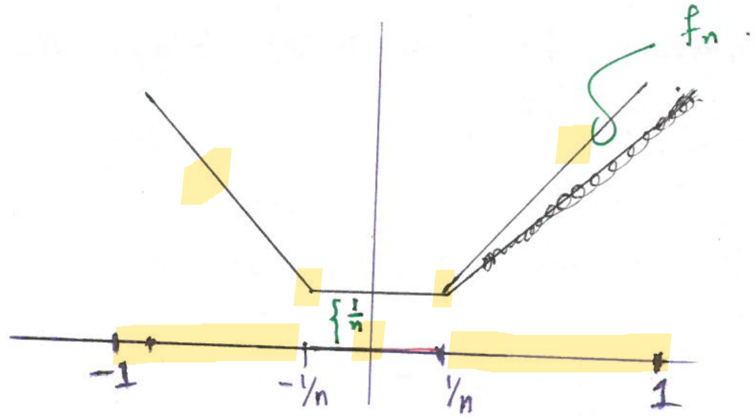
$|x_n - x| < \varepsilon \;\; \forall n \geq N$

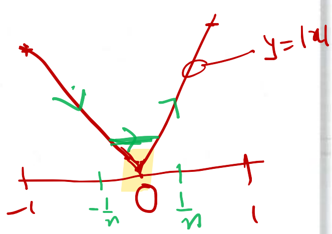
Now this "looks" like modulus.

Ensures that $f_n - f \in \mathcal{B}(S)$ $\forall \; n \geq N$.

eg: Let $S = [-1, 1]$. Define $f_n \in \mathcal{F}(S)$ by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq \frac{1}{n}. \\ |x| & \text{if } \frac{1}{n} < |x| \leq 1. \end{cases}$$





Then $\lim_{n \to \infty} f_n(x) = |x| \quad \forall \; x \in S.$

i.e. $f_n \xrightarrow{p} f$ on S, where

$$f(x) = |x| \quad \forall \; x \in S.$$

$\forall$ fixed $x \in [-1, 1]$

$$\left[\; |f_n(x) - f(x)| = \begin{cases} \left|\frac{1}{n_0} - |x|\right| & \text{if } |x| \leq 1/n \\ & \text{if } 1/n < |x| \leq 1 \end{cases} \Rightarrow f_n(x) \to f(x) \;\; \forall \, x \right]$$

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In fact: $\|f_n - f\| = \sup_{x\in[-1,1]} |f_n(x) - f(x)|$

$= \frac{1}{n} \quad \forall n$. $\leftarrow$ $\because$ the max occurs at $x=0$.

$\therefore$ For $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t.

$$\|f_n - f\| < \varepsilon \quad \forall n \geq N.$$

$$\Rightarrow f_n \xrightarrow{u} f \text{ on } [-1,1].$$

$\boxed{///}$

Remark: $f_n \xrightarrow{u} f \Rightarrow f_n \xrightarrow{p} f$. (i.e. unif $\Rightarrow$ point.)

Proof: Suppose that $f_n \longrightarrow f$ uniformly on S.

Let $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ $\ni$.

$$\|f_n - f\| < \varepsilon \quad \forall n \geq N.$$

i.e. $\sup_x |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$,

i.e. $|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, x \in S$.

$\Rightarrow f_n(x) \longrightarrow f(x) \quad \forall x \in S$

$\therefore f_n \longrightarrow f$ pointwise. $\boxed{///}$

\# $\therefore$ For u.c. it is perhaps a good idea to compute the pointwise limit (if exists) first!!

Although computation of pointwise limit may be diff. occasionally.

[If pointwise limit fails to exists, then u.c. also fails.]

Remark

\# "$\Leftarrow$" of the above is NOT true! i.e. pointwise $\not\Rightarrow$ unif.

eg: $f_n(x) = x^n$ on $[0,1]$. $\leftarrow$ $\{f_n\}$ is NOT u.c. (HW).

\# Now, we develop some useful tools for convergency!!

~~The above examples suggests a pattern!~~

Thm: (Cauchy' criterion) : Let $\{f_n\} \subseteq \mathcal{F}(S)$.

~~$\lim_{n\to\infty} f_n(x) = f(x) \ \forall x \in S$.~~ [~~i.e. $f_n \xrightarrow{p} f$~~].

(This is a must for u.c.)

Then $\{f_n\}$ is u.c. ~~Therefore $f_n \xrightarrow{u}$~~ $\iff$ for $\varepsilon > 0 \ \exists \ N \in \mathbb{N}$ s.t.

$$\|f_m - f_n\| < \varepsilon \quad \forall \ m, n \geq N.$$

(Also ensures that $f_m - f_n \in \mathcal{B}(S) \ \forall m, n$).

No need to worry about limit fn. f.

Proof: "$\Rightarrow$" Let $f_n \xrightarrow{u} f$, and let $\varepsilon > 0$.

$\therefore \ \exists \ N \in \mathbb{N}$ s.t. $\|f_n - f\| < \varepsilon/2 \quad \forall \ n \geq N$.

$\therefore \ \forall \ m, n \geq N$, we have:

$$\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\|$$

$$\leq \|f_n - f\| + \|f_m - f\|$$

Δ-ineq.

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Standard proof Like for $\{x_n\} \subseteq \mathbb{R}$

"$\Leftarrow$" Let $\varepsilon > 0$.

$\therefore \ \exists \ N \in \mathbb{N} \ \ni$ $\underbrace{\|f_n - f_m\|}_{= \sup_{x \in S} |f_n(x) - f_m(x)|} < \varepsilon/2 \quad \forall \ m, n \geq N$.

~~i.e. $\{f_n(x)\}_{n\geq 1}$ is Cauchy $\forall x \in S$.~~

i.e. $|f_n(x) - f_m(x)| < \varepsilon/2 \quad \forall \ m, n \geq N \ \& \ x \in S$. — (*)

$\Rightarrow \{f_n(x)\}$ is Cauchy $\forall x \in S$. $\Rightarrow \lim_{n\to\infty} f_n(x) := f(x)$ exists $\forall x \in S$.

i.e. $f_n \xrightarrow{p} f$.

Also (*) $\Rightarrow f_n(x) - \varepsilon/2 < f_m(x) < f_n(x) + \varepsilon/2$ $\quad \forall m, n \geq N$, $x \in S$.

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$f(x) := \lim_{n\to\infty} f_n(x)$.

But $\lim_{m\to\infty} f_m(x) = f(x) \quad \forall\, x \in S$.

$\therefore$ Taking limit as $m \to \infty$, we have.

$$f_n(x) - \varepsilon/2 \leq f(x) \leq f_n(x) + \varepsilon/2 \quad \forall\, n \geq N \;\&\; x \in S.$$

$$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon \quad \forall\, n \geq N \;\&\; x \in S$$

$$\Rightarrow \|f_n - f\| < \varepsilon \quad \forall\, n \geq N.$$

i.e. $f_n \xrightarrow{u} f$ on S. ▨

Cauchy Criterion is often useful.

eg: Let $\{r_n\}_{n=1}^{\infty}$ $= \{f(n) : n \in \mathbb{N}\}$. be an enumeration of rationals $\mathbb{Q} \cap [0,1]$.

$f : \mathbb{N} \xrightarrow{bij} \mathbb{Q} \cap [0,1]$

$\# \, \mathbb{Q} \cap [0,1] = \{r_n\}$

For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x = r_1, \dots, r_n \\ 1 & \text{if } x \neq r_1, \dots, r_n. \end{cases}$$

$y = 1$

$r_1 \cdots r_n$

Then $\{f_n\}$ is not u.c. on $[0,1]$.

Indeed, Choose $\varepsilon = \frac{1}{2}$. ~~Then~~

Observe, for each $n \in \mathbb{N}$, ~~$f_m(r_m)$ = ...~~

we have $f_n(r_{n+1}) = 1$ & $f_{n+1}(r_{n+1}) = 0$.

$$\Rightarrow |f_n(r_{n+1}) - f_{n+1}(r_{n+1})| = 1 \quad \forall\, n.$$

$\therefore \nexists\, N \in \mathbb{N}$ s.t. $|f_m(x) - f_n(x)| < \varepsilon = \frac{1}{2}$ $\forall\, m, n \geq N$.

$\because$ By Cauchy Criterion, $\{f_n\}$ is NOT u.c.

[Q: $\{f_n\}$ Converges pointwise?] ▨

Thm: (M-test): Let $\{f_n\} \subseteq \mathcal{F}(S)$ & suppose $f_n \xrightarrow{p} f$.

Set $M_n := \sup_{x \in S} |f_n(x) - f(x)|$.

Then $f_n \xrightarrow{u} f$ on $S \iff M_n \longrightarrow 0$.

Proof: "$\Rightarrow$" Let $f_n \xrightarrow{u} f$, & let $\varepsilon > 0$.

$\therefore \exists\, N \in \mathbb{N} \ni \underbrace{\|f_n - f\|}_{= M_n} < \varepsilon \quad \forall\, n \geq N$.

$= M_n$. ← By def.

i.e $M_n < \varepsilon \quad \forall\, n \geq N$.

$\Rightarrow M_n \longrightarrow 0$.

"$\Leftarrow$" Let $\varepsilon > 0$. Then $\exists\, N \in \mathbb{N} \ni$

$M_n < \varepsilon \quad \forall\, n \geq N$.

$\Rightarrow \|f_n - f\| < \varepsilon \quad \forall\, n \geq N$.

$\Rightarrow f_n \xrightarrow{u} f$. ☒

$\frac{nx}{1+n^2x^2} = \frac{x}{\frac{1}{n}+nx^2} \to 0$ as $n \to \infty$ ✓ as x is fixed.

eg: $\forall\, n \in \mathbb{N}$, $x \in [0,1]$, define

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

$f_n(x) < 1 \quad \forall\, n$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

Note that: $\cancel{nx < 1+n^2x^2}$ $\forall\, n$ & fixed $x \in [0,1]$.

$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x)$ i.e. $f_n \xrightarrow{p} f$,

Fix $x \in [0,1]$, $\lim_{n\to\infty} f(x) =$

where $f(x) \equiv 0$. ✓

$\therefore M_n = \|f_n - f\| = \sup_{x \in [0,1]} \left[\frac{nx}{1+n^2x^2}\right] \quad \forall\, n$.

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But, $x = 0 \Rightarrow f_n(x) = 0 \quad \forall n.$

& $x > 0 \Rightarrow \dfrac{\frac{1}{nx} + nx}{2} \geqslant \sqrt{\frac{1}{nx} \times nx} = 1.$

$$\Rightarrow \frac{1}{2} \times \frac{1+n^2x^2}{nx} \geqslant 1.$$

$$\Rightarrow \frac{nx}{1+n^2x^2} \leqslant \frac{1}{2}.$$

And, "=" occurs at $x = \frac{1}{n}$.

$\therefore \; M_n = \frac{1}{2} \quad \forall \; n \geqslant 1.$

$\Rightarrow M_n \not\rightarrow 0.$

$\therefore$ By M-test, $\{f_n\}$ is NOT unif. convergent on $[0,1]$.

$\boxed{//}$

eg: $\forall n \in \mathbb{N}$, define $f_n(x) = x^n(1-x). \quad x \in [0,1].$

~~$f_n(x) =$~~ $\left\{ \begin{array}{l} \therefore f_n(1) = 0 = f_n(0) \quad \forall n. \\ \& \text{ for } 0 < x < 1, \quad f_n(x) \rightarrow 0. \end{array} \right.$

$\therefore \; f_n \xrightarrow{p} 0$ the zero fn.

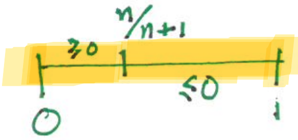
$\therefore \; M_n = \sup\limits_{x \in [0,1]} \left[x^n(1-x) \right].$

But $f_n'(x) = x^{n-1}\left[n - (n+1)x \right].$

$f'\left(\frac{n}{n+1}\right) = 0$

$\therefore \; f_n'(x) \geqslant 0 \quad \forall \; x \in \left[0, \frac{n}{n+1}\right]$

& $f_n'(x) \leqslant 0 \quad \forall \; x \in \left[\frac{n}{n+1}, 1\right]$



$\because$ $f_n(0) = f_n(1) = 0$, it follows that f_n has a max. at $x = \frac{n}{n+1}$.

Now $f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \times \frac{1}{n+1} < \frac{1}{n+1}$.

i.e. $M_n < \frac{1}{n+1}$.

$\Rightarrow M_n \longrightarrow 0$.

$\therefore$ By M-test, $\{f_n\}$ is u.c. $\left(\& \ f_n \xrightarrow{u} 0\right)$. $\boxtimes$

~~eg Consider $S = \mathbb{R}$.~~

Remark: Suppose $|f_n(x) - f(x)| \leq M_n \quad \forall\, x \in S,\ n \geq 1$.

all n but finitely many.

If $M_n \to 0$, then $f_n \xrightarrow{u} f$ on S.

Proof: Follows from M-test.

eg: Let $r > 0$ & $f_n(x) := e^{-nx} \quad \forall\, x \in [r, \infty)$.

Now $f_n \downarrow$ on $\mathbb{R}$ $\forall n$.

$\Rightarrow \quad 0 < f_n(x) = e^{-nx} \leq e^{-nr} \quad \forall\ n \geq 1\ \&\ x \geq r$.

But $e^{-nr} \longrightarrow 0$ as $n \longrightarrow \infty$.

$\Rightarrow f_n \longrightarrow 0$ uniformly on $[r, \infty)$. $\boxtimes$