Jaydeb Sarkar.

Compatibility :- (Examples) :

eg: Recall $\{f_n\} \subseteq C[0,1]$, where $f_n(x) = x^n$; $x \in [0,1]$, $n \in \mathbb{N}$.

Then $f_n \xrightarrow{p} f$, where $f(x) = \begin{cases} 0 & x \in [0,1). \\ 1 & x = 1. \end{cases}$

Clearly, $f \notin C[0,1]$.

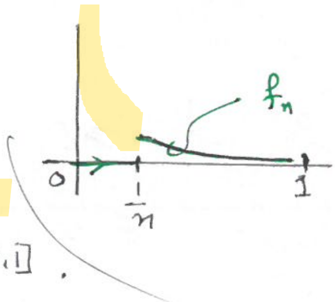
$\therefore$ ~~C[0,1]~~ pointwise Convergency & continuity are not compatible.

[Q: If $\{f_n\}$ are cont. & $f_n \xrightarrow{u} f$ on S
$\overset{?}{\Longrightarrow}$ f is cont ?] Also, if f_n is cont. at

[x_0 $\forall n$ ✓, does $\lim_{n\to\infty} \underbrace{\lim_{x\to x_0} f_n}_{f_n(x_0)} = \lim_{x\to x_0} \underbrace{\lim_{n\to\infty} f_n}_{= f}$ $\left(= \lim_{x\to x_0} f\right)$?] $= f(x_0)$.

eg: $f_n(x) = \begin{cases} 0 & x \in [0, 1/n] \\ \frac{1}{x} & x \in [0,1] \setminus [0, 1/n] \end{cases}$

$\therefore \{f_n\} \subseteq \mathcal{B}[0,1]$.



However, $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \in (0,1] \end{cases}$.

$\Rightarrow \lim_{n\to\infty} f_n(x) := f(x) \quad \forall x \in [0,1]$

is NOT bdd.

$\therefore$ pointwise Convergency does not preserve boundedness !!

Q: What about u.c ?

If $f_n \xrightarrow{u} f$
$\Rightarrow f_n \xrightarrow{p} f$

$\mathbb{Q} \cap [0,1] = \{r_n\}_{n=1}^{\infty}$

eg: Consider an enumeration $\{r_n\}_{n=1}^{\infty}$ of rationals $\mathbb{Q} \cap [0,1]$.

Define $f_n(x) = \begin{cases} 0 & \text{if } x = r_1, \dots, r_n \\ 1 & \text{if } x \in [0,1] \setminus \{r_1, \dots, r_n\}. \end{cases}$

$\therefore f_n \in \mathcal{R}[0,1] \quad \forall n.$ [$\because f_n$ is discont. at finitely many points. $\forall n$.]

Now for $m \in \mathbb{N}$, we know: $f_n(r_m) = 0 \quad \forall n \geq m.$

$\Rightarrow f_n(r_m) \to 0$ as $n \to \infty$. $\forall m \in \mathbb{N}$.

$\therefore \forall x \in \{r_n\}_{n=1}^{\infty}, \quad \lim_{n\to\infty} f_n(x) = 0.$

Next, let $x \in [0,1] \setminus \mathbb{Q}$.

$\therefore f_n(x) = 1 \ \forall n \Rightarrow \lim_{n\to\infty} f_n(x) = 1.$

$\therefore f_n \xrightarrow{p} f$ on $[0,1]$,

where $f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c. \end{cases}$

But we know that $f \notin \mathcal{R}[0,1]$.

$\therefore \lim_{n\to\infty} f_n(x) \notin \mathcal{R}[0,1]$.

$\therefore \mathcal{R}[a,b]$ is Not closed under pointwise convergency!!

[Q: What if $f_n \to f$ unif. ?]

Even if $\lim f_n = f \in \mathcal{R}[a,b]$ $\overset{?}{\Longrightarrow} \int \lim f_n = \lim \int f_n$?

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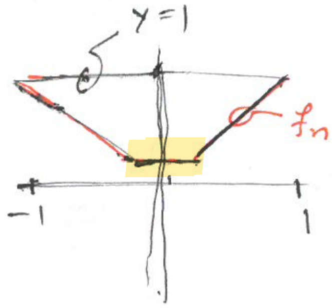
eg: Recall that $f_n \to f$ uniformly on $[-1,1]$, where

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq \frac{1}{n}. \\ |x| & \text{if } \frac{1}{n} < |x| \leq 1. \end{cases}$$



& $f(x) = |x|$, $x \in [-1,1]$.

Note that f_n is diff. at 0 $\forall n$.

However, f is NOT diff. at 0.

Here the situation is even worse: as $f_n \longrightarrow f$ unif. on $[-1,1]$.

$\Downarrow$

[U.C is not compatible with diff!!]

All the examples yield negative feeling about the following compatibility issue:

Fact Suppose $\{f_n\} \subseteq \mathcal{F}(S)$, $f \in \mathcal{F}(S)$. Suppose $f_n \longrightarrow f$ $\boxed{\text{pointwise}}$ on S.

Let f_n is cont. on S $\forall n$. $\overset{?}{\Longrightarrow}$ f is cont. on S ? NO!

Let $f_n \in \mathcal{B}(S)$ $\forall n$. $\overset{?}{\Longrightarrow}$ $f \in \mathcal{B}(S)$? NO!

Let $f_n \in \mathcal{R}[a,b]$. $\forall n$. $\overset{?}{\Longrightarrow}$ $f \in \mathcal{R}[a,b]$? NO!

If so, then must it be true that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n \quad \left(= \int_a^b f \right) \quad ? \leftarrow ??$$

(pending)

Let f_n is diff. at $x \in S$, $\forall n$. $\overset{?}{\Longrightarrow}$ f' exists at x? NO!

If so, then must it be true that

$$\lim_{n\to\infty} f_n'(x) = f'(x) \quad ? \quad \leftarrow ??$$

(pending)

\# Suppose $\lim_{x \to x_0} f_n$ exists $\forall\, n$. $\overset{?}{\Longrightarrow}$ $\lim_{x \to x_0} f$ exist? NO!

If so, is it true that

$$\lim_{n \to \infty} \left(\lim_{x \to x_0} f_n \right) = \lim_{x \to x_0} f \quad ?$$

~~Pending~~ NO

eg: See Page-13

eg: $f_n(x) = \begin{cases} 0 & 0 < x \le 1/n \\ \frac{1}{x} & 1/n < x \le 1 \end{cases}$ $f_n : (0,1] \to \mathbb{R}$.

$\Rightarrow \lim_{x \to 0} f_n$ exists $\forall\, n$. But $\lim_{x \to 0} \Big(\underbrace{\lim f_n}_{= \frac{1}{x}} \Big)$ DNE !!

eg: Consider $f_n(x) = x^n$, $x \in (0,1)$.

We know, $\lim_{n \to \infty} f_n(x) = f(x)$ $\forall\, x$, where $f \equiv 0$.

i.e. $f_n \to 0$ pointwise on $(0,1)$.

Now, $\lim_{x \to 1} f_n(x) = 1$ $\forall\, n$.

But $\lim_{x \to 1} f(x) = 0$.

$$\therefore \lim_{n \to \infty} \lim_{x \to 1} f_n(x) \neq \lim_{x \to 1} \lim_{n \to \infty} f_n(x).$$

— × —

All in all:

Pointwise Convergence is a natural concept $\boxed{\text{but}}$ with a number of disadvantages !!

AND, Indeed, one would like to capture all the above properties of Convergence !!

Jaydeb Sarkar

In the following, we prove that with uniform convergency, all problems disappear.

BUT, NOT ~~with~~ for differentiability !! ⟵ We will work this out too!!

Thm: Let $x_0 \in S$ & $f_n \xrightarrow{u} f$ on $S \setminus \{x_0\}$. If $\lim_{x \to x_0} f_n$ exists $\forall n$, then $\lim_{x \to x_0} f$ also exists. In this case,

$$\lim_{n \to \infty} \lim_{x \to x_0} f_n = \lim_{x \to x_0} f.$$

[i.e. $\lim_{n \to \infty} \lim_{x \to x_0} f_n = \lim_{x \to x_0} \lim_{n \to \infty} f_n$.] ∴ Interchange of limits.

U.C & limit

Proof: Let $\varepsilon > 0$. Since $f_n \xrightarrow{u} f$, by Cauchy criterion, $\exists\, N \in \mathbb{N}$ s.t.

$$\|f_n - f_m\| < \varepsilon/2 \quad \forall\, n, m \geq N. \text{ ON } S \setminus \{x_0\} \qquad (*)$$

$\forall\, n \in \mathbb{N}$, set $a_n := \lim_{x \to x_0} f_n$.

Now, $a_n - a_m = \lim_{x \to x_0} [f_n(x) - f_m(x)]$ $\forall\, n, m \geq 1$.

$$\Rightarrow |a_n - a_m| = \lim_{x \to x_0} |f_n(x) - f_m(x)| \qquad [\because\ ||a| - |b|| \leq |a - b|]$$

$$\leq \frac{\varepsilon}{2} \quad [\text{by } (*)] \quad \forall\, m, n \geq N$$

$\Rightarrow |a_n - a_m| \leq \varepsilon/2 \quad \forall\, m, n \geq N$.

$\Rightarrow \{a_n\}$ is Cauchy.

$\therefore \exists\, a \in \mathbb{R}$ s.t. $a := \lim_{n \to \infty} a_n$.

$$\boxed{\therefore\ a = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x).}$$

Again, $f_n \xrightarrow{u} f$ on $S\setminus\{x_0\}$ gives: $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\|f_n - f\| < \varepsilon/3 \quad \forall\, n \geq n_0, \text{ on } S\setminus\{x_0\}. \qquad \text{(i)}$$

Also, for $a_n \to a$, $\exists\, \tilde{n}_0 \in \mathbb{N}$ s.t.

$$|a_n - a| < \varepsilon/3 \quad \forall\, n \geq \tilde{n}_0. \qquad \text{(ii)}$$

Set $\hat{n} := \max\{n_0, \tilde{n}_0\}$. Focus is on $\hat{n}$ now!!

$\because \lim_{x\to x_0} f_{\hat{n}} = a_{\hat{n}}$, $\exists\, \delta > 0$ s.t.

$$|f_{\hat{n}}(x) - a_{\hat{n}}| < \varepsilon/3 \quad \forall\, x \in S\setminus\{x_0\} \text{ s.t. } |x - x_0| < \delta. \qquad \text{(iii)}$$

$\therefore$ for each $x \in S\setminus\{x_0\}$ & $|x-x_0| < \delta$, we have:

$$|f(x) - a| \leq |f(x) - f_{\hat{n}}(x)| + |f_{\hat{n}}(x) - a_{\hat{n}}| + |a_{\hat{n}} - a|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad (\text{by (i)} - \text{(iii)})$$

$$= \varepsilon.$$

Typical $\varepsilon/3$-argument.

$\therefore\ |f(x) - a| < \varepsilon \quad \forall\, x \in S\setminus\{x_0\}$ & $|x - x_0| < \delta$.

$$\Rightarrow \lim_{x\to x_0} f = a.$$

i.e. $$\lim_{x\to x_0} f = \lim_{n\to\infty} \lim_{x\to x_0} f_n.$$

$\boxtimes$

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unif. Cont. & limit

Thm: (Continuity) Let $f_n \xrightarrow{u} f$ on S. Let $x_0 \in S$ & let each f_n is continuous at x_0. Then f is also cont. at x_0.

Proof: We know $\lim_{x \to x_0} f_n = f_n(x_0) \quad \forall n$. [$\because f_n$ is cont. at x_0].

Also, $f_n \xrightarrow{u} f \implies \lim_{n\to\infty} f_n(x_0) = f(x_0)$.

$$\therefore f(x_0) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} \lim_{x\to x_0} f_n$$

$$= \lim_{x\to x_0} \lim_{n\to\infty} f_n \quad \text{[by previous thm.]}$$

$$= \lim_{x\to x_0} f$$

$\implies$ f is cont. at x_0. ▨

Thm: (Bounded fn's). Let $\{f_n\} \subseteq \mathcal{B}(S)$ & $f_n \xrightarrow{u} f$ on S. Then $f \in \mathcal{B}(S)$.

u.c & bdd

[$\because \mathcal{B}(S)$ is closed under uniform limits.]

Proof: $\because f_n \xrightarrow{u} f$ on S, for $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t.

$$\|f_n - f\| < 1 \quad \forall n \geq N.$$

Then, $\forall x \in S$, we have:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$$

$$< 1 + \|f_N\|.$$

$$\implies \|f\| < 1 + \|f_N\| \implies f \in \mathcal{B}(S).$$

▨

Thm: (Riemann integration) Let $\{f_n\} \subseteq \mathcal{R}[a,b]$ & let $f_n \xrightarrow{u} f$ on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ & $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

u.c & integ.

$$\left[\text{i.e. } \lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n\right].$$

Interchange of limits.

Proof: Let $\varepsilon > 0$. Since $f_n \xrightarrow{u} f$ on $[a,b]$, $\exists\ N \in \mathbb{N}$ s.t.

$$\|f_n - f\| < \frac{\varepsilon}{b-a} \quad \forall\ n \geq N. \qquad \text{(i)}$$

Focus on N.

In particular: $\|f_N - f\| < \dfrac{\varepsilon}{b-a}$. i.e. $\sup_{x\in[a,b]} |f_N(x) - f(x)| < \dfrac{\varepsilon}{b-a}$.

$$\text{i.e. } f(x) - \frac{\varepsilon}{b-a} < f_N(x) < f(x) + \frac{\varepsilon}{b-a}. \qquad \text{(ii)}$$

$\because f_N \in \mathcal{R}[a,b]$, $\exists\ P \in \mathcal{P}[a,b]$ s.t.

$$U(f_N, P) - L(f_N, P) < \varepsilon. \qquad \text{(iii)}$$

Now (ii) $\Rightarrow f(x) < f_N(x) + \dfrac{\varepsilon}{b-a} \quad \forall\ x \in [a,b]$

$\Rightarrow U(f, P) < U(f_N, P) + \varepsilon$.

Illy (ii) $\Rightarrow f(x) > f_N(x) - \dfrac{\varepsilon}{b-a} \quad \forall\ x \in [a,b]$

$\Rightarrow L(f, P) > L(f_N, P) - \varepsilon$.

So, (iii) $\Rightarrow U(f_N, P) - L(f_N, P) < \varepsilon$

$\Rightarrow \big(U(f,P) - \varepsilon\big) - \big(L(f,P) + \varepsilon\big) < \varepsilon$.

$\Rightarrow U(f, P) - L(f, P) < 3\varepsilon$.

$\Rightarrow f \in \mathcal{R}[a,b]$.

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Finally, we prove that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

$\forall\, x \in [a,b]$, we have

$$\left| \int_a^x f - \int_a^x f_n \right| = \left| \int_a^x (f_n - f) \right| \leq \int_a^x |f_n - f| .$$

$\therefore$ By (i), $\forall\, n \geqslant N$, we have:

$$\left| \int_a^x f - \int_a^x f_n \right| \leq \frac{\varepsilon}{b-a} \times \int_a^x 1 \, \cancel{dx} = \frac{\varepsilon}{b-a} \times (x-a)$$

$$\leq \varepsilon \qquad \forall\, x \in [a,b].$$

$\therefore \left\{ \int_a^x f_n \right\}$ converges unif. to $\int_a^x f$ on $[a,b]$.

In particular: $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. $\boxtimes$

In fact: We proved that:

$$\lim_{n\to\infty} \int_a^x f_n = \int_a^x \lim f_n \qquad \forall\, x \in [a,b].$$