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Finally, we prove that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

$\forall\, x \in [a,b]$, we have

$$\left| \int_a^x f - \int_a^x f_n \right| = \left| \int_a^x (f_n - f) \right| \leq \int_a^x |f_n - f| .$$

$\therefore$ By (i), $\forall\, n \geqslant N$, we have:

$$\left| \int_a^x f - \int_a^x f_n \right| \leqslant \frac{\varepsilon}{b-a} \times \int_a^x 1 = \frac{\varepsilon}{b-a} \times (x-a).$$

$$\leqslant \varepsilon \qquad \forall\, x \in [a,b].$$

$\therefore$ $\left\{ \int_a^x f_n \right\}$ Converges unif. to $\int_a^x f$ on $[a,b]$.

In particular: $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

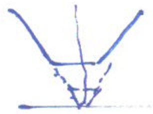
$\square$

In fact: We proved that:

$$\lim_{n\to\infty} \int_a^x f_n = \int_a^x \lim f_n \qquad \forall\, x \in [a,b].$$

u.c. & differentiability:

Recall: $f_n \longrightarrow f$ & $f_n'(x_0)$ exists $\forall n$ $\not\Rightarrow$ $f'(x_0)$ exists.

{eg: 

Another extreme example:

eg: Suppose $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in \mathbb{R}$, $n \geq 1$.

$\therefore \{f_n\} \subseteq C(\mathbb{R})$,

& $$|f_n(x)| = \left|\frac{x}{1+nx^2}\right|$$

$$\leq \frac{|x|}{2\sqrt{n}\,|x|} \quad \forall x \neq 0$$

$\left[\because 1+nx^2 \geq 2\sqrt{n}|x| \quad \forall x \in \mathbb{R} \Rightarrow \dfrac{1}{1+nx^2} \leq \dfrac{1}{2}\dfrac{1}{\sqrt{n}|x|} \quad \forall x \in \mathbb{R}, n \geq 1, x \neq 0.\right]$

$$= \frac{1}{2\sqrt{n}} \quad \forall x \neq 0.$$

~~since.~~

$$\therefore \quad |f_n(x)| \leq \frac{1}{2\sqrt{n}} \quad \forall n \geq 1, x \in \mathbb{R}.$$

$(\because f_n(0) = 0 \ \forall n)$.

$\Rightarrow f_n \xrightarrow{u} f$, where $f \equiv 0$ on $\mathbb{R}$.

M-test.

[Note that $f_n(0) = 0 \ \forall n \geq 1$]

Now: $$f_n'(x) = \frac{-nx^2+1}{(1+nx^2)^2}$$ $\left[-x \times 2nx + (1+nx^2) = -nx^2+1\right]$

$$= \frac{-nx^2+1}{(1+nx^2)^2} \quad \forall x \ \cancel{\neq 0}.$$

[Note: $f_n'(0) = 1 \ \forall n$

$\because \dfrac{1-n}{(1+n)^2} = \dfrac{1/n - 1}{(1/n^2 + 1/n)^2} \longrightarrow 0$ as $n \longrightarrow \infty$,

we have $f_n'(x) \longrightarrow F(x) \ \forall x \in \mathbb{R}$,

where $$F(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

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In particular, $f_n'(0) = 1 \quad \forall n$

$$\Rightarrow \lim_{n\to\infty} f_n'(0) = 1 \neq f'(0) = 0.$$

i.e. although ~~even~~ ~~$\{f_n\}$~~ $f_n \xrightarrow{u} f$ & $\{f_n\}$ & $\{f\}$ diff. at x_0, still

$$\frac{d}{dx}\left(\lim_{n\to\infty} f_n\right) \neq \lim_{n\to\infty}\left(\frac{d}{dx} f_n\right). \;!!$$

Failure of interchange of limits.

— x —

Still, we have some news:

Thm: Let $\{f_n\}$ be a seqn of diff. fn on $[a,b]$ s.t.

(i) $\exists\, x_0 \in [a,b]$ s.t. $\{f_n(x_0)\}$ converges.

(ii) $\{f_n'\}$ converges uniformly on $[a,b]$.

~~(iii)~~ Then $\{f_n\}$ converges uniformly on $[a,b]$ to a diff. fn. f &

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad \forall x \in [a,b].$$

Proof: Let $\varepsilon > 0$. $\exists\, N \in \mathbb{N}$ s.t.

$$\left.\begin{aligned} \|f_n' - f_m'\| &< \frac{\varepsilon}{2(b-a)} \\ \& \quad |f_n(x_0) - f_m(x_0)| &< \frac{\varepsilon}{2} \end{aligned}\right\} \forall\, m,n \geq N. \quad \longrightarrow (*)$$

Fix $m,n \geq N$,

& fix $x \in [a,b] \setminus \{x_0\}$. By MVT ($\because f_n - f_m$ is diff.)

(applied to $f_n - f_m$)

$\exists\, \xi$ between x & x_0 s.t.

$$(f_n - f_m)(x) - (f_n - f_m)(x_0) = (f_n - f_m)'(\xi) \times (x - x_0).$$

$$\Rightarrow f_n(x) - f_m(x) = \left(f_n(x_0) - f_m(x_0)\right) + \left(f_n'(\xi) - f_m'(\xi)\right)(x - x_0).$$

$$\Rightarrow \quad |f_n(x) - f_m(x)| \underset{\Delta\text{-ineq.}}{\leqslant} |f_n(x_0) - f_m(x_0)| + |x - x_0|\,|f_n'(\xi) - f_m'(\xi)|$$

$$\underset{\text{by} (*)}{<} \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} \times |x - x_0|$$

$$\leqslant \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \left[\because |x - x_0| \leq b - a\right]$$

$$\Rightarrow \quad |f_n(x) - f_m(x)| < \varepsilon \qquad \forall\ n, m \geqslant N \ \&\ x \in [a,b].$$

(For $x = x_0$ it is true anyway).

$(**)$

i.e $\|f_n - f_m\| < \varepsilon \quad \forall\ m, n \geqslant N.$

$\Rightarrow$ $\{f_n\}$ is u.c. on $[a,b]$.

Let $\lim_{n\to\infty} f_n = f$ on $[a,b]$.

$\boxed{\text{Claim:}}$ f is diff. on $[a,b]$ & $f'(x) = \lim_{n\to\infty} f_n'(x)$ $\forall\ x \in [a,b]$.

~~Fix $\tilde{x} \in [a,b]$. Set~~ $\longrightarrow$ $\longrightarrow$ Next page $\longrightarrow$

$$\tilde{f}_n(x) := \begin{cases} \dfrac{f_n(x) - f_n(\tilde{x})}{x - \tilde{x}} & \forall\ x \neq \tilde{x} \\ f_n'(\tilde{x}) & \text{if } x = \tilde{x}. \end{cases}$$

$$\& \quad \tilde{f}(x) = \begin{cases} \dfrac{f(x) - f(\tilde{x})}{x - \tilde{x}} & \forall\ x \neq \tilde{x} \\ f \end{cases}$$

Fix $\tilde{x} \in [a,b]$.

Set $F_n(x) := \dfrac{f_n(x) - f_n(\tilde{x})}{x - \tilde{x}}$

& $F(x) := \dfrac{f(x) - f(\tilde{x})}{x - \tilde{x}}$ } $\forall n$ & $x \neq \tilde{x}$.

$\because f_n$ is diff. on $[a,b]$ $\forall n$, it follows that

$$\lim_{x \to \tilde{x}} F_n(x) = f_n'(\tilde{x}) \quad \forall n. \quad (*)$$

Also, for each $x \in [a,b] \setminus \{\tilde{x}\}$, by MVT, $\exists\ \xi$ between x & $\tilde{x}$

s.t. $(f_n - f_m)(x) - (f_n - f_m)(\tilde{x}) = (x - \tilde{x}) \times (f_n - f_m)'(\xi)$.

LHS $= (f_n(x) - f_n(\tilde{x})) - (f_m(x) - f_m(\tilde{x}))$

$\Rightarrow \left(F_n(x) - F_m(x) \right) = f_n'(\xi) - f_m'(\xi)$.

(Recall: $\| f_n' - f_m' \| < \varepsilon / 2(b-a)$).

$\therefore (*)$ implies, $\quad |F_n(x) - F_m(x)| < \dfrac{\varepsilon}{2(b-a)} \quad \forall m, n \geq N$ & $x \neq \tilde{x}$.

$\Rightarrow \| F_n - F_m \| < \dfrac{\varepsilon}{2(b-a)} \quad \forall n, m \geq N$ on $[a,b] \setminus \{\tilde{x}\}$

$\Rightarrow \{F_n\}$ is u.c. on $[a,b] \setminus \{\tilde{x}\}$.

$\Rightarrow \quad \lim_{n \to \infty} \underbrace{\lim_{x \to \tilde{x}} F_n(x)}_{= f_n'(\tilde{x})} = \lim_{x \to \tilde{x}} \underbrace{\lim_{n \to \infty} F_n(x)}$.

interchanging limits.

But $f_n \xrightarrow{u} f \Rightarrow F_n(x) \longrightarrow \dfrac{f(x) - f(\tilde{x})}{x - \tilde{x}} = F(x)$ as $n \to \infty$

i.e. $\lim_{n \to \infty} F_n(x) = F(x)$, $\forall x \neq \tilde{x}$.

$\therefore \lim_{n\to\infty} \lim_{x\to\tilde{x}} F_n(x) = \lim_{x\to\tilde{x}} F(x)$.

$\therefore$ (✡) $\Rightarrow \lim_{n\to\infty} f_n'(\tilde{x}) = \lim_{x\to\tilde{x}} F(x)$. — (†)

$\because F(x) = \dfrac{f(x)-f(\tilde{x})}{x-\tilde{x}}$ & since the LHS of (†) ~~limit~~ exists,

it follows that $\lim_{x\to\tilde{x}} F(x) = f'(\tilde{x})$ exists.

$$\therefore \quad f'(\tilde{x}) = \lim_{n\to\infty} f_n'(\tilde{x}). \qquad \boxed{//}$$

Another exotic example:

$\{f_n\}$ be a diff. fn. & $f_n \longrightarrow f$ unif.

Suppose f is also diff. Still, $f_n' \not\longrightarrow f'$ in general !!

[just see the previous eg.]

eg: $f_n(x) = \dfrac{\sin nx}{n}$ $x \in [0,1]$, $n \geq 1$.

$\because \left|\dfrac{\sin nx}{n}\right| \leq \dfrac{1}{n}$, $\forall n$, it follows

that $f_n \longrightarrow f$ unif. on $[0,1]$, where

$f \equiv 0$.

$\therefore f' \equiv 0$. on $[0,1]$. Also, $f_n'(x) = \cos nx$.

$\forall n$, & $x \in [0,1]$.

But $f_n'(x)$ does not converge for $x \in (0,1]$.

$\therefore f_n' \not\longrightarrow f'$. (~~even point~~ $\lim_{n\to\infty} f_n'(x)$ DNE. $\boxed{//}$

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Series of functions.

Def. Let $\{f_n\} \subseteq \mathcal{F}(S)$. The formal sum of functions

$$f_1 + f_2 + \cdots \quad := \sum_{n \geq 1} f_n$$

is called a series of f_n's.

$\forall x \in S$, by $\sum f_n(x)$, we understand the formal sum $f_1(x) + f_2(x) + \cdots$

Given $\sum_{n=1}^{\infty} f_n$, $\forall n \in \mathbb{N}$, define the n-th partial sum of the series as:

$$S_n(x) = \sum_{k=1}^{n} f_n(x). \qquad (\forall x \in S). \quad \cancel{n \in \mathbb{N}}.$$

(Finite sum : So good to go.)

Def: Let $f : S \to \mathbb{R}$ & $\{f_n\}_{n \geq 1} \subseteq \mathcal{F}(S)$. We say that

f is the pointwise limit of $\sum f_n$ if $S_n \to f$ pointwise on S. In this case, we write $\sum f_n = f$, pointwise.

If $S_n \xrightarrow{u} f$ on S, then we say that $\sum f_n = f$ uniformly on S. $\boxtimes$

eg: $S = [0, 1)$. $f_n(x) = x^n$ $\forall$ n, $x \in [0, 1)$.

$$\therefore S_n(x) = \sum_{m=0}^{n-1} x^m \implies S_n(x) = \frac{1 - x^n}{1 - x} \quad \forall x \in [0, 1).$$

$$\therefore \lim_{n \to \infty} S_n(x) = \frac{1}{1 - x} \quad \forall x \in [0, 1).$$

i.e. $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$ pointwise on $[0, 1)$.

However, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ is $\boxed{\text{not}}$ uniform on $[0,1)$.

Indeed: $\frac{1}{1-x} \notin \mathcal{B}[0,1)$. However:

$$S_n \in \mathcal{B}[0,1) \quad \& \quad \lim_{n\to\infty} S_n(x) = \frac{1}{1-x}.$$

▨

Thm: (Cauchy Criterion) Let $\{f_n\} \subseteq \mathcal{F}(S)$. Then $\sum f_n$ converges uniformly on S $\Longleftrightarrow$ for $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t.

$$\left\| \sum_{k=m+1}^{n} f_k \right\| < \varepsilon \qquad \forall\ n > m \geq N.$$

Proof: $\because$ $S_n = \sum_{k=1}^{n} f_k$, it follows that

$$S_n - S_m = \sum_{k=m+1}^{n} f_k \qquad \forall\ n > m.$$

Then $\sum f_n$ is u.c. $\Longleftrightarrow$ $\{S_n\}$ is u.c. $\Longleftrightarrow$ Cauchy criterion for seq. of fn's. ▨

Cor: If $\sum f_n$ converges uniformly on S, then $\|f_n\| \to 0$ as $n \to \infty$.

Proof: $\because$ $f_n = S_{n+1} - S_n$ $\forall n$, this follows from the above theorem. ▨

$f_n(x) = \frac{1}{n}\ \forall x \in \mathbb{R}$ ✗

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eg: Consider $0 < \varepsilon < 1$ & $\sum_{n=0}^{\infty} x^n$. Let $n > m$, $x \in [-\varepsilon, \varepsilon]$.

Now $|S_n(x) - S_m(x)| = \frac{1}{|1-x|} \times |(1-x^n) - (1-x^m)|$

$$= \frac{|x^n - x^m|}{|1-x|} \leq \frac{2|x|^m}{|1-x|} \qquad \left(\because n > m \ \& \ |x| < 1\right)$$