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$\therefore \; |S_n(x) - S_m(x)| \leq \dfrac{2|x|^m}{|1-x|} \leq \dfrac{2|x|^m}{1-|x|}$ $\quad n>m$

$|x|<\varepsilon \Rightarrow -|x| > -\varepsilon$
$\Rightarrow 1-|x| > 1-\varepsilon$
$\Rightarrow \dfrac{1}{1-|x|} < \dfrac{1}{1-\varepsilon}$

$\leq \dfrac{2\varepsilon^m}{1-\varepsilon}$ $\quad (\because x \in [-\varepsilon,\varepsilon] \;\&\; 0<\varepsilon<1.)$

i.e. $|S_n(x) - S_m(x)| \leq 2 \times \dfrac{\varepsilon^m}{1-\varepsilon}$ $\quad \forall x \in [-\varepsilon,\varepsilon]$ & $n>m$.

$\Leftrightarrow \left\| \sum_{k=m+1}^{n} f_k \right\| \leq 2\dfrac{\varepsilon^m}{1-\varepsilon} \quad \forall n>m.$

$\because \varepsilon^m \to 0$ as $m \to \infty$, for $\tilde{\varepsilon}>0 \;\exists N \in \mathbb{N}$ s.t. $2\dfrac{\varepsilon^m}{1-\varepsilon} < \tilde{\varepsilon} \quad \forall m \geq N.$

$\because 0<\varepsilon<1$

$\therefore \forall n > m \geq N$

$\left\| \sum_{k=m+1}^{n} f_k \right\| < \tilde{\varepsilon}$

by Cauchy Criterion

$\Rightarrow \sum_{n=0}^{\infty} x^n$ is u.c. on $[-\varepsilon,\varepsilon]$ $\forall\; 0<\varepsilon<1$.

" # $\therefore \sum_{n=0}^{\infty} x^n$ is u.c. on all compact subsets of $(-1,1)$. "

Of Course: $\boxed{\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}}$ $\forall x \in [-\varepsilon,\varepsilon]$.

$\mathbb{R}\setminus\{1\}$

If $\sum f_n$ ~~u.c.~~ on S, then it is u.c. on $\tilde{S}$, $\forall \tilde{S} \subseteq S$.

There are other ways to prove the above Conclusion.

— x —

Thm: Let $\{f_n\} \subseteq \mathcal{F}(S)$, $\|f_n\| \leq M_n$ $\forall n \geq 1$ & Suppose $\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum f_n$ Converges uniformly & absolutely on S.

M-test

$\|f_n\| = \sup_{x \in S} |f_n(x)|.$

[**Def:** $\sum f_n$ is absolutely convergent if $\sum |f_n(x)|$ converges $\forall x$.]

Proof: $\because \|f_n\| < M_n \Rightarrow |f_n(x)| < M_n \ \forall x \ \& \ n$,

by Comparison test, $\sum f_n$ is absolutely convergent.

Now, $\forall n > m$, we have

$$\|S_n - S_m\| = \left\|\sum_{k=m+1}^{n} f_k\right\| \leq \sum_{k=m+1}^{n} \|f_k\| \leqslant \sum_{k=m+1}^{n} M_k .$$

$\because \sum_{n=1}^{\infty} M_n < \infty$, by Cauchy criterion, $\sum f_n$ is u.c. $\boxtimes$

 # eg: ① $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ $\quad p > 1$, $x \in \mathbb{R}$.

Here $f_n(x) := \dfrac{\sin nx}{n^p}$ $\forall n$, $x \in \mathbb{R}$.

$\therefore \|f_n\| \leq \dfrac{1}{n^p}$.

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $(p>1)$, by Weierstass M-test,

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ converges absolutely & uniformly on $\mathbb{R}$.

② Set $f_n(x) = \dfrac{x}{n+n^2x^2}$ $(x \in \mathbb{R})$. $\therefore f_n(0) = 0 \ \forall n$.

For $x \neq 0$; $|f_n(x)| = \dfrac{|x|}{n+n^2x^2} = \dfrac{1}{\underbrace{\frac{n}{|x|} + n^2|x|}_{\geqslant 2n^{3/2}}} \leqslant \dfrac{1}{2n^{3/2}}$

$\Rightarrow \|f_n\| \leqslant \dfrac{1}{2n^{3/2}}$.

$\therefore$ By Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n+n^2x^2}$ is absolutely & u.c. on $\mathbb{R}$.

$\boxtimes$

(3) u.c but NOT absolutely Convergent : (Not so easy example. But we also have almost a trivial one : wait).

useful technique.

Consider $\sum f_n$ on $\mathbb{R}$, with

$$f_n(x) = \frac{(-1)^{n+1}}{n+x^2} \quad \forall x \in \mathbb{R}.$$

i.e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+x^2}$ on $\mathbb{R}$.

Now for each fixed $x \in \mathbb{R}$, $|f_n(x)| = \frac{1}{n+x^2}$.

But $\sum_{n=1}^{\infty} \frac{1}{n+x^2}$ is divergent. [Note: For $x \in \mathbb{R}$ fixed, given $n \in \mathbb{N}$ $\exists$ $m \in \mathbb{N}$ s.t. $n+x^2 < n+m$ $\Rightarrow \frac{1}{n+x^2} > \frac{1}{n+m}$]

$\Rightarrow$ $\sum f_n$ is not absolutely convergent $\forall x \in \mathbb{R}$.

Now we prove that $\sum f_n$ is u.c.

$\forall n \in \mathbb{N}$, observe that :

$$S_{2n}(x) = \left(\frac{1}{1+x^2} - \frac{1}{2+x^2}\right) + \left(\frac{1}{3+x^2} - \frac{1}{4+x^2}\right) + \cdots + \left(\frac{1}{2n-1+x^2} - \frac{1}{2n+x^2}\right) \quad \forall x \in \mathbb{R}.$$

$\because$ all terms are > 0, it follows that $S_{2n}(x) \uparrow$.

Also, $\sum f_n(x)$ is an an alternating series & easy to see it converges $\forall x \in \mathbb{R}$. ← Why ?

Set $f(x) := \sum f_n(x) \quad \forall x \in \mathbb{R}$.

$\sum f_n(x) = \lim_{n\to\infty} S_n(x)$

$\therefore$ $f(x) - S_{2n}(x) > 0 \quad \forall x$.

Also, $f(x) - S_{2n}(x) = \frac{1}{2n+1+x^2} - (+\text{ve no.})$.

$f(x) - S_{2n+1}(x) < 0$

$$< \frac{1}{2n+1+x^2} < \frac{1}{2n} \quad \forall x \in \mathbb{R}.$$

i.e. $f(x) - S_{2n}(x) < \frac{1}{2n}$ $\quad \forall\, x \in \mathbb{R},\ n \in \mathbb{N}$.

$f_n(x) = \frac{(-1)^{n+1}}{n+x^2}$

$S_{2n+1}(x) - f(x) = S_{2n}(x) - f(x) + f_{2n+1}(x)$, where $f_{2n+1}(x) > 0$

Illy $\quad S_{2n+1}(x) - f(x) > 0$

& $S_{2n+1}(x) - f(x) < \frac{1}{2n}$.

$\therefore \forall\, n \in \mathbb{N}$ & $x \in \mathbb{R}$,

$$\left.\begin{aligned} 0 &< f(x) - S_{2n}(x) < \tfrac{1}{2n} \\ \& \quad 0 &< S_{2n+1}(x) - f(x) < \tfrac{1}{2n} \end{aligned}\right\} \Downarrow \quad \Rightarrow \|S_n - f\| < \tfrac{1}{2n} \ \forall n.$$

i.e. $|S_n(x) - f(x)| < \frac{1}{2n}$ $\quad \forall\, n$ & $x \in \mathbb{R}$.

$\Rightarrow \quad S_n \xrightarrow{u} f$ on $\mathbb{R}$

$\therefore \sum_{n=1}^{\infty} f_n$ is u.c on $\mathbb{R}$. $\quad \boxtimes$

Even Simpler : $f_n(x) = \frac{(-1)^{n+1}}{n}$ $\forall\, n, x$. Then $\sum f_n$ is u.c but NOT A.C !!

Thm: (Limit - Series thm.) Suppose $\sum f_n = f$ uniformly on $S \setminus \{x_0\}$ for some $x_0 \in S$. If $\lim_{x \to x_0} f_n$ exists $\forall\, n \in \mathbb{N}$, then

$\sum_{n=1}^{\infty} \lim_{x \to x_0} f_n$ ~~exist~~ Converges &

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n .$$ Interchange of limits.

Proof: For $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $\left\| \sum_{k=m+1}^{n} f_k \right\| < \varepsilon/2$ i.e.

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon/2 \qquad \forall\ n > m \geq N \ \& \ x \in S \setminus \{x_0\}$$

$\because \lim_{x \to x_0} f_k$ exists $\forall\, k$, & as the above sum is finite, it follows that

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$$\left| \sum_{k=m+1}^{n} \lim_{x\to x_0} f_k \right| < \varepsilon \quad \forall\ n > m \geq N.$$

$$\Rightarrow \alpha := \sum_{n=1}^{\infty} \lim_{x\to x_0} f_n \text{ Converges.}$$

You can start the proof from right HERE!

{F_n} on $S\setminus\{x_0\}$. Let $F_n \to F$ unif on $S\setminus\{x_0\}$. If $\lim_{x\to x_0} F_n$ exists then $\lim_{x\to x_0} F$ also exists ~~we have~~ & $\lim_{x\to x_0} F(x)$ exists $\forall n$. $= \lim_{n\to\infty} \lim_{x\to x_0} F_n(x)$

~~Finally~~, since $\sum f_n = f$ unif. on $S\setminus\{x_0\}$,

$S_n \longrightarrow f$ uniformly on $S\setminus\{x_0\}$.

$\because$ $S_n = \sum_{k=1}^{n} f_k$ & $\lim_{x\to x_0} f_n$ exists $\forall n$,

the limit $\lim_{x\to x_0} S_n = \sum_{k=1}^{n} \lim_{x\to x_0} f_k$

$\therefore$ By "limit – u.c. thm", it follows that

$S_n = F_n$

$$\left[\lim_{x\to x_0} S_n\right] \xrightarrow{n\to\infty} \lim_{x\to x_0} f \quad \text{as } n \to \infty.$$

i.e. $\displaystyle \lim_{n\to\infty}\left(\lim_{x\to x_0} \sum_{k=1}^{n} f_k\right) = \lim_{x\to x_0} f$

i.e. $\displaystyle \cancel{\lim_{n\to\infty}} \sum_{n=1}^{\infty} \lim_{x\to x_0} f_n = \lim_{x\to x_0} \sum f_n$. $\boxtimes$

Illy using partial sums & corresponding results in u.c. we have the following.

\# Let $\sum f_n = f$ unif. on S. If f_n is bdd $\forall n$, then f is also bdd.

\# Let $\sum f_n = f$ unif. on $[a,b]$, $f_n \in \mathcal{R}[a,b]$ $\forall n$.

Then $f \in \mathcal{R}[a,b]$ & $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$.

\# If $f_n \in C(S)$, $\forall n$, & $\sum f_n$ Converges unif. then

$\sum f_n \in C(S)$.

\# Illy derivatives. — HW —

Similar proof.

$\boxed{\text{Illy}}$

~~Thm: (Dini's thm on u.c) Let $S \subseteq \mathbb{R}$ be compact, $\{f_n\} \subseteq C(S)$ & $f_n \longrightarrow f \in C(S)$ pointwise. If $\{f_n\}$ is monotonic~~~~ally~~ ~~decreasing~~ (i.e. $\{f_n(x)\} \downarrow$ or $\uparrow$ $\forall x \in S$), then~~

~~$f_n \longrightarrow f$ uniformly on S.~~

~~Proof: If possible, let $f_n \not\to f$ unif. on S.~~

~~$\therefore \exists \; \varepsilon > 0$ s.t. $\underbrace{\|f_n - f\|}$ $> \varepsilon$ for infinitely many $n \in \mathbb{N}$.~~

~~$\|f_n - f\| = \sup_{x \in S} \{|f_n(x) - f(x)|\}$.~~

~~$= \sup_{x \in S} \{ f_n(x) - f(x) \}$~~

~~$[\because f_n(x) \downarrow f(x) \;\forall x \in S]$.~~

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Thm: (Dini's theorem on u.c.).

Let $S \subseteq \mathbb{R}$ be compact, $\{f_n\} \subseteq C(S)$ & let $f_n \longrightarrow f \in C(S)$ pointwise. If $\{f_n\}$ is monotonic (i.e. $\{f_n(x)\} \downarrow$ or $\uparrow$ $\forall x \in S$) then $f_n \longrightarrow f$ uniformly on S.

For ~~series~~ seqn of f_n's: $\{f_n\}$ we know if $\lim f_n = f$ unif. & f_n cont., then f is cont. This is a "kind" of converse.

Proof: WLOG: assume $f_n \downarrow$ i.e.

$$f_n(x) \geq f_{n+1}(x) \quad \forall x \in S,\ n \geq 1.$$

Set $F_n = f_n - f \quad \forall n.$ $\left(\because f_n \xrightarrow{p} f\right)$

$\therefore \{F_n\} \subseteq C(S)$, $F_n \downarrow$, $F_n \geq 0$.

i.e. $F_n(x) \geq F_{n+1}(x) \geq 0 \quad \forall x \in S,\ n \geq 1.$

Recall $\sup\{F_n(x) : x \in S\} = \|F_n\|$, $\forall n$.

Claim: $\|F_n\| \longrightarrow 0$.

Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, define

$$O_n := F_n^{-1}(-\infty, \varepsilon) \quad \left(= \{x \in S : F_n(x) < \varepsilon\}\right)$$

$\because F_n \in C(S)$, we have that O_n open in S $\forall n$.

Also $F_n \downarrow \Rightarrow O_{n+1} \supseteq O_n \quad \forall n$.

$\because f_n(x) \longrightarrow f(x) \quad \forall x \in S$, it follows that

$$F_n(x) \longrightarrow 0 \quad \forall x \in S.$$

$\therefore$ For each $x \in S$, $\exists N_x \in \mathbb{N}$ s.t.

$$F_{N_x}(x) < \varepsilon.$$

$\Rightarrow x \in O_{N_x}$.

$\therefore$ For $x \in S$, $\exists N \in \mathbb{N}$ s.t. $x \in O_N$.

$\Rightarrow \bigcup_{n=1}^{\infty} O_n = S$. $\therefore \{O_n\}$ an open cover of S.

$O_{n_1}, O_{n_2}, \dots, O_{n_K}$, $n_K = N$

But S is compact.

$\Rightarrow \exists\, N \in \mathbb{N}$ s.t. $\bigcup_{n=1}^{N} O_n = S$.

i.e. $\boxed{O_N = S.}$ $\leftarrow \because O_n \uparrow$.

i.e. $\{x \in S : F_N(x) < \varepsilon\} = S$.

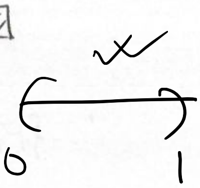
$\Rightarrow F_N(x) < \varepsilon \quad \forall\, x \in S$.

$\Rightarrow \|F_N\| < \varepsilon$.

$\because \varepsilon > 0$ is arb. it follows . . .

But $F_n \downarrow \;\Rightarrow\; \|F_n\| \longrightarrow 0$. ▨

$(\because \|F_n\| \leq \|F_N\| < \varepsilon \;\; \forall\, n \geq N)$.



Remark:

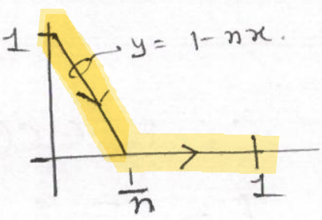
(1) S ~~is~~ compact is necessary:

$f_n(x) = x^n$ on $(0,1)$.

$\therefore f_n \downarrow$ & $f_n \xrightarrow{p} 0$. But $f_n \not\to 0$ unif. on $(0,1)$.

(2) Continuity of f $(= \lim f_n$, pointwise$)$ ~~is continuous~~ is also necessary:

$$f_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq 1/n \\ 0 & 1/n < x \leq 1. \end{cases}$$



$\frac{x}{1/n} + \frac{y}{1} = 1$

$\Rightarrow y = 1 - nx$

That's the idea.

$\therefore f_n \xrightarrow{p} f$

where $f(x) = \begin{cases} 1 & x = 0. \\ 0 & x \in (0,1]. \end{cases}$

$\therefore f \notin C[0,1]$.

& $f_n \not\to f$ unif. as $\|f_n - f\| = 1 \;\; \forall\, n$.

$\Rightarrow \|f_n - f\| \not\to 0$. ▨

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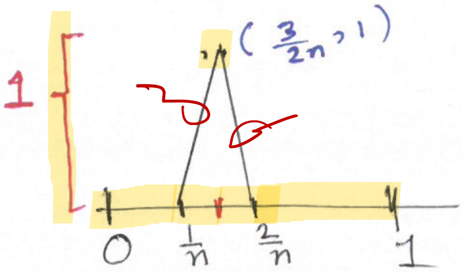
(3) monotonicity of f_n ~~monotonic~~ is also necessary:

Define $f_n : [0,1] \longrightarrow \mathbb{R}$ by



$f_n(x) = \{$??

$\therefore$ $f_n \in C[0,1]$ & f_n not monotone.

Also $f_n \longrightarrow 0$ pointwise but

$\|f_n\| = 1 \ \forall n \implies f_n \not\to 0$ unif. ☒

— x —.