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Then, with $g_n(x) = \frac{1}{n}$, $x \in [\varepsilon, 2\pi-\varepsilon]$, we conclude by the (full) Dirichlet test, that

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx \quad \text{converges uniformly on } [\varepsilon, 2\pi-\varepsilon].$$

$$\forall\ 0<\varepsilon<2\pi.$$

$\boxed{//}$

eg: Let $p>1$. Then, by M-test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx$$

converges uniformly to some fn. $f(x)$ on all of $\mathbb{R}$.

$\because$ the convergence is unif. on $\mathbb{R}$, ~~in~~ & since

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \qquad (x \in \mathbb{R}),$$

by term-by-term integration:

$$\int_0^{\pi/2} f = \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \right) dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^p} \int_0^{\pi/2} \cos nx \, dx.$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \sin\left(\frac{n\pi}{2}\right).$$

$$\Rightarrow \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{p+1}}. \quad \forall p>1.$$

Next, Consider $p \in (0,1]$.

Then, as in the previous example, use the Dirichlet test to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \quad \text{converges unif. on } [\varepsilon, 2\pi-\varepsilon]$$
$$\forall \varepsilon \in (0, 2\pi)$$
$$\& \quad p \in (0,1].$$

$\boxed{///}$

eg: Let $0 < \varepsilon < 1$. We know:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{uniformly on } [-\varepsilon, \varepsilon].$$

$\therefore$ $\forall x \in [-\varepsilon, \varepsilon]$, we have:

$$\int_0^x \left(\sum_{n=0}^{\infty} t^n \right) dt = \int_0^x \frac{1}{1-t}\, dt.$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\int_0^x t^n\, dt \right) = -\log(1-x).$$

← BTW: What is the defn. of log?

$$\Rightarrow -\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$

$$\Rightarrow \log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \forall x \in [-\varepsilon, \varepsilon].$$

Thus: $\boxed{\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}} \quad \forall x \in (-1, 1)$

& the convergency is unif. on $[-\varepsilon, \varepsilon]$

$\forall \; 0 < \varepsilon < 1$.

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eg: Note that:

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \qquad \forall\ x \in (-1,1).$$

uniformly on all compact subsets of $(-1,1)$.

$\therefore\ \forall\ x \in (-1,1)$, we have:

why?

$$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n}\right) dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt.$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$

$$\Rightarrow \boxed{\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}} \qquad \forall\ x \in (-1,1)$$

& unif. on $[-\varepsilon, \varepsilon]$ $\forall\ 0 < \varepsilon < 1$.

$\blacksquare$

In particular: if $x = \frac{1}{2}$ & $x = \frac{1}{3}$, then

$$\underbrace{\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3}}_{\parallel} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}}\right) \times \frac{(-1)^n}{2n+1}$$

$$\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \tan^{-1} 1 = \pi/4.$$

$$\Rightarrow \pi = 4 \times \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}}\right).$$

$\blacksquare$

eg: (Derivatives).

Consider the series: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right)$.

$x=0 \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ — Alternating series. & converges.

$\therefore \{f_n(x_0)\}$ converges for $x_0 = 0$.

Also, $\displaystyle\frac{d}{dx}\left(\frac{(-1)^{n-1}}{n} \cos\frac{n}{x}\right) = \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{n}{x}\right)$

Now by M-test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{n}{x}\right)$$

— $\therefore f_n'$ conv. unif. on $\mathbb{R}$.

Converges unif. on $\mathbb{R}$.

$$\therefore \quad f(x) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right)$$

defines a diff. fn. in $\mathbb{R}$. & the above series converges unif. on $\mathbb{R}$. Moreover:

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{n}{x}\right) \quad \forall x \in \mathbb{R}.$$

Also, u.c.

∎

§ Sequences of improper integrals:

Let ~~$f_n, f \in$~~ $\{f_n\} \subseteq \mathcal{R}[a,\infty)$

& let $f_n \longrightarrow f$ or $\sum f_n = f$

pointwise or uniformly.

Q: $\displaystyle\lim_{n\to\infty} \int_a^{\infty} f_n = \int_a^{\infty} f$ or $\displaystyle\sum \int_a^{\infty} f_n = \int_a^{\infty} f$??

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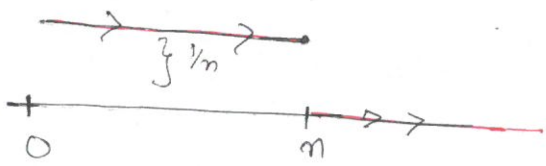
Of course, we need the assurance of $f \in \mathcal{R}[a, \infty)$!!

Note: perhaps with pointwise convergence, there is no/less hope.
What about u.c. ??

$\Big\}$ Again, no hope.

eg: $\forall n \in \mathbb{N}$, define $f_n : [0, \infty) \longrightarrow \mathbb{R}$ by:



i.e. $$f_n(x) = \begin{cases} \frac{1}{n} & 0 \leq x \leq n \\ 0 & x > n. \end{cases}$$

Clearly, $f_n \in \mathcal{R}[0, \infty)$ & $\int_0^\infty f_n = 1 \quad \forall n.$

Also, $\lim_{n \to \infty} f_n(x) = 0 \quad \forall x \in [0, \infty)$ & uniformly.

$\therefore \quad \lim_{n \to \infty} \int_0^\infty f_n = 1 \neq \int_0^\infty \lim_{n \to \infty} f_n.$ $\quad \left[\because \|f_n\| \leq \frac{1}{n}\right]$

Observe that $\nexists\, g \in \mathcal{R}[0, \infty)$ s.t.

$$f_n(x) \leq g(x) \quad \forall x \in [0, \infty).$$

And this lack of dominance is the key:

For instance (a baby version of dominated convergence thm).

Thm (DCT) ⟵ Very useful theorem !!

Let $\{f_n\} \subseteq C[a,\infty)$ s.t. $f_n \to f$ unif. on $[a,b]$ $\forall\, b > a$.

Let $g \in C[a,\infty) \cap \mathcal{R}[a,\infty)$ s.t.

$$|f_n(x)| \leq g(x) \quad \forall\, x \in [a,\infty).$$

Then $$\lim_{n\to\infty} \int_a^\infty f_n = \int_a^\infty f.$$

Proof: HW.

---

§ Power Series.

Def: Let $c \in \mathbb{R}$ & $\{a_n\}_{n\geq 0} \subseteq \mathbb{R}$. The formal sum

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{—— (P)}$$

is called a power series about c (or with center c) with coefficients $\{a_n\}$.

eg: Polynomials are power series & admit (P) for any $c \in \mathbb{R}$! Q: Given a polynomial, how you write it as (P)?

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Note:

(1) Setting $f_n(x) = a_n(x-c)^n$, we realize that the P.S. (P) is perhaps the simplest series of fn's
$$\sum_{n=0}^{\infty} f_n(x).$$

(2) Our interest: Convergence (pointwise / absolute / uniform) of the series (P) & then determine the limit fn.

(3) Adopt all the ideas of series of fn's.

(4) If $S \subseteq \mathbb{R}$ s.t. (P) converges $\forall\, x \in S$, then $\exists\, f : S \to \mathbb{R}$ s.t.
$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \forall\, x \in S.$$

Clearly, $S \neq \varphi$ as $c \in S$!

(5) eg: $\sum_{n=0}^{\infty} x^n$ is a P.S. with center at 0 & coefficients $\{1\}$.

We know: $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$; ~~$x \in [0,1)$~~ $x \in (-1,1)$

Geometric series.

$\frac{1}{1-x} = f(x)$

Defined only on $(-1,1)$.

Defined on $\mathbb{R} \setminus \{1\}$

$\therefore \{P.S\} \longleftrightarrow \{\text{Functions}\}$

We need to understand this link (if any).
& Corresponding domain of definitions !!

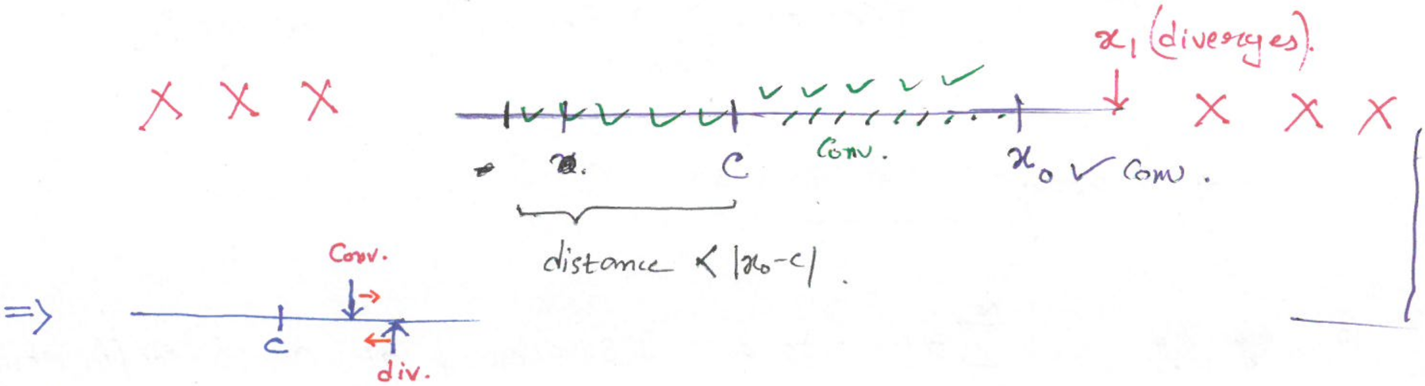
The following is remarkable:

Thm: Consider the P.S. $\sum_{n=0}^{\infty} a_n (x-c)^n$ — Ⓟ.

① If Ⓟ converges for some $x_0 \in \mathbb{R}\setminus\{c\}$, then it converges absolutely for all x s.t. $|x-c| < |x_0 - c|$.

② If Ⓟ diverges at $x_1 \in \mathbb{R}$, then it diverges $\forall\, x \in \mathbb{R}$ s.t. $|x-c| > |x_1 - c|$.

$\therefore$ We have the following picture:



Proof: Let Ⓟ converges at $x = x_0$. (Also, $x_0 \neq c$).

$$\Rightarrow \sum_{n=0}^{\infty} a_n (x_0 - c)^n \text{ converges.}$$

A series of real no's.

$$\Rightarrow a_n (x_0 - c)^n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

$\therefore$ For $\varepsilon = \frac{1}{2}$, $\exists\, N \in \mathbb{N}$ s.t.

$$|a_n (x_0 - c)^n| < \frac{1}{2} \quad \forall\, n \geq N.$$

Let $x \in \mathbb{R}$ & suppose $|x - c| < |x_0 - c|$.

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$$\therefore \ |a_n (x-c)^n| = |a_n (x_0-c)^n| \times \left|\frac{x-c}{x_0-c}\right|^n$$

$$\underset{\forall n \geq N}{<} \frac{1}{2} \times \left|\frac{x-c}{x_0-c}\right|^n$$

But $|x-c| < |x_0-c| \Rightarrow \left|\frac{x-c}{x_0-c}\right| = r$ for some $r \in (0,1)$.

$\therefore \ |a_n(x-x_0)^n| < \frac{1}{2} r^n \quad \forall \ n \geq N$.

$\therefore$ By Comparison test, $\sum_{n=0}^{\infty} |a_n(x-x_0)|^n$ converges.

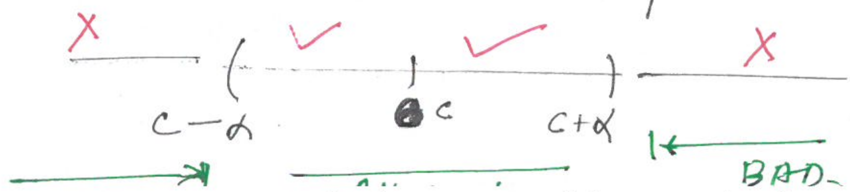
i.e. (P) is A.C. $\forall \ x \in \mathbb{R}$ s.t. $|x-c| < |x-x_0|$.

For (2): Let $x \in \mathbb{R}$ s.t. $|x-c| > |x_1-c|$.

If (P) converges at x, then by (1), (P) will converge at x_1 $\rightarrow\leftarrow$. $\boxtimes$

\# The above result is curious: We need to find the "maximum" of x_0 s.t. $\sum a_n(x-x_0)^n$ converges & " min. of x_1 s.t. $\sum a_n(x-x_1)^n$ diverges.

If there is such a "min-max", say α, then



Q: How to find/determine that "min"max" ?

Recall: (Root test)

Let $\sum_{n=0}^{\infty} \alpha_n$ be a series of +ve no's.

Set $r = \limsup \sqrt[n]{\alpha_n}$.   possibly, $R = +\infty$.

① If $r < 1$, then the series converges.

② If $r > 1$, then the series diverges.

③ If $r = 1$, the test is inconclusive.

— HW —.

$\boxed{\text{CONVENTION}}$: $\frac{1}{\infty} = 0$; $\frac{1}{0} = \infty$.

Thm: (Cauchy – Hadamard thm).

Consider the P.S. $\sum_{n=0}^{\infty} a_n(x-c)^n$. & set

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}.$$

Then the P.S converges $\forall x \in \mathbb{R}$ s.t. $|x-c| < R$.

& diverges $\forall x \in \mathbb{R}$ s.t. $|x-c| > R$.

Proof: Observe that $\forall x \in \mathbb{R}$,

$$\limsup |a_n(x-c)^n|^{1/n} = \frac{|x-c|}{R}.$$

∴ The result follows from the Root test.

☒