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Def: Given a P.S. $\sum_{n=0}^{\infty} a_n (x-c)^n$, the number $R \in \mathbb{R} \cup \{\infty\}$ is called the <u>radius of convergence</u>, where

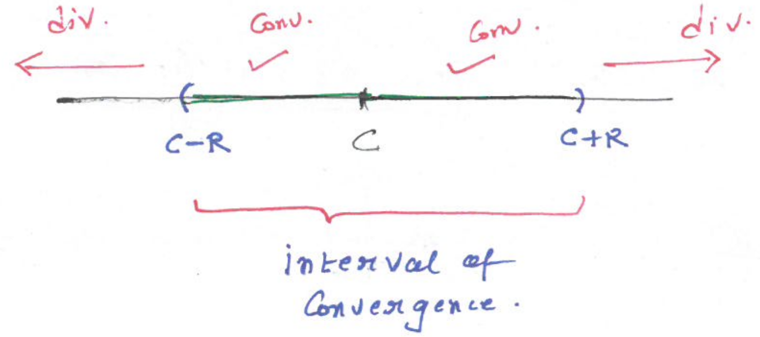
$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|} .$$

Recall: $\frac{1}{\infty} = 0$ & $\frac{1}{0} = \infty$

Moreover, $(c-R, c+R)$ is called the <u>interval of convergence of the P.S.</u>

\# $\therefore$ By Cauchy-Hadamard thm., for $\sum_{n=0}^{\infty} a_n (x-c)^n$ & with

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|} ,$$



* No conclusion about end points $\{c \pm R\}$.

\# If $R = 0$, then the series converges only at $x = c$.

Remark: Let $\{\alpha_n\}$ be a seqn. of +ve no's. Then: HW

$$\liminf \frac{\alpha_{n+1}}{\alpha_n} \leq \liminf \sqrt[n]{\alpha_n} \leq \limsup \sqrt[n]{\alpha_n} \leq \limsup \frac{\alpha_{n+1}}{\alpha_n}$$

$\therefore$ $\boxed{\text{If } \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \text{ exists}}$, then $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \lim_{n\to\infty} \sqrt[n]{\alpha_n}$.

Cor: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then for $\sum_{n=0}^{\infty} a_n (x-c)^n$, the radius of convergence is given by:

useful for computation.

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

eg: ① Let $p \in \mathbb{R}[x]$. Fix $c \in \mathbb{R}$.

$\Rightarrow p = \sum_{n=0}^{N} a_n (x-c)^n$. ——— ⊛ for some $N \in \mathbb{N}$.

$\therefore \frac{1}{R} = \limsup \sqrt[n]{|a_n|} = 0$

$\Rightarrow R = \infty$.

$\therefore$ the r.o.c. (radius of convergence) of ⊛ is ∞.

$\Rightarrow \mathbb{R}$ is the interval of convergence.

& the P.S. conv. absolutely on $\mathbb{R}$.

② $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. ⟵ $c = 0$, $a_n = \frac{1}{n!}$ $\therefore$ A P.S.

$\therefore \frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$.

$\Rightarrow R = \infty$.

$\therefore$ r.o.c, i.e, $R = \infty$ & $\mathbb{R}$ is the interval of conv.

We define: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\forall x \in \mathbb{R}$.

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(3) $\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots$

$$a_n = \begin{cases} \frac{1}{3} & n = 0. \\ \frac{1}{3^n} & n \text{ even.} \\ -1 & n \text{ odd.} \end{cases} \Rightarrow \limsup \sqrt[n]{|a_n|} = 1.$$

$$\Downarrow$$

$$R = 1.$$

$\therefore$

$\therefore$ Radius of Conv. is 1.

(4) $$\sum_{n=0}^{\infty} \frac{n!}{(n+1)^{n+1}} x^n \longleftarrow$$ A P.S. about 0. with Coefficients $\left\{\frac{n!}{(n+1)^{n+1}}\right\}$.

Here $c = 0$, $a_n = \frac{n!}{(n+1)^{n+1}}$.

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \frac{n!}{(n+1)!} \times \frac{(n+2)^{n+2}}{(n+1)^{n+1}}$$

$$= \frac{1}{n+1} \times \frac{(n+2)^{n+2}}{(n+1)^{n+1}} = \left(\frac{n+2}{n+1}\right)^{n+2}$$

$$= \left(1 + \frac{1}{n+1}\right)^{n+2} \longrightarrow e \text{ as } n \to \infty.$$

why ??

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e}.$$

$\Rightarrow$ Radius of Convergence is $= e$.

$\therefore$ The P.S. Converges absolutely on $\{x \in \mathbb{R} : |x| < e\}$.

(5) $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n . \qquad \left(= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right).$$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

$\Rightarrow$ $R = 1$.

$\therefore$ $(-1, 1)$ is the interval of convergency.

However (END POINTS): If $x = 1$, we have $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which is convergent.

If $x = -1$, then we have $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ ← divergent.

[$\because$ end points are ~~alwa~~ not determined : rather it depends case by case.].

— x —

Remark: WLOG: we may simply study P.S. with center 0.

$\therefore$ From now on: $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.

Recall: If $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges at $x_0 \in \mathbb{R}$ $(x_0 \neq 0)$, then $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x \in \mathbb{R}$ s.t. $|x| < |x_0|$.

The idea was: $\sum_{n=0}^{\infty} a_n x^n$ converges at x_0

$$\Rightarrow \sum_{n=0}^{\infty} a_n x_0^n \text{ conv.}$$

$$\Rightarrow a_n x_0^n \longrightarrow 0 \quad \Rightarrow \text{ for } \varepsilon = \tfrac{1}{2},\ \exists\, N \in \mathbb{N} \text{ s.t.}$$

$$|a_n x_0^n| < \tfrac{1}{2} \quad \forall n \geq N.$$

Now if $|x| < |x_0|$, then

$$|a_n x^n| < |a_n x_0^n| \quad \forall n \geq 1.$$

$$\cancel{\Rightarrow \sum_{n=0}^{\infty} |a_n x^n| < \sum_{n=0}^{\infty} |a_n x_0^n|}$$

$\because \sum |a_n|\,|x_0^n| < \infty$, $\qquad x$-~~free. &~~ $< \infty$.

By M-test, $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-x_0, x_0]$.

Thus, we have the following:

\# **Thm:** Let R = radius of convergence of the P.S. $\sum_{n=0}^{\infty} a_n x^n$.

Then $\sum_{n=0}^{\infty} a_n x^n$ is $\boxed{\text{u.c}}$ on all closed intervals $\subseteq (-R, R)$.

Very useful property.

The following is now easy:

Thm: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. — the radius of convergence. Assume $R \neq 0$.

Then $\forall x \in (-R, R)$,

$$\int_0^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

on compact subsets.

Proof: $\because$ term-by-term int. is allowed for u.c. series.

Q: What about <u>derivatives of P.S.</u>?

R = radius of convergence, always.

Notation: ~~Remark~~: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$.

$$\left(\therefore 1/R = \limsup \sqrt[n]{|a_n|} \right)$$

Def: Given a P.S. $\sum_{n=0}^{\infty} a_n x^n$, the <u>derived series</u> is the new P.S. $\sum_{n=1}^{\infty} n a_n x^{n-1}$ ← The term-by-term derivatives.

Thm: Let R_d = radius of convergence of the derived P.S. $\sum_{n=1}^{\infty} n a_n x^{n-1}$.

Then $R = R_d$.

$\left[R = 1/\limsup \sqrt[n]{|a_n|} \right]$.

Proof: By definition:

$$1/R_d = \limsup \underbrace{\sqrt[n]{n|a_n|}}_{\sqrt[n]{n} \times \sqrt[n]{|a_n|}}$$

& we know $\sqrt[n]{n} \to 1$ as $n \to \infty$.

$$\therefore \limsup \sqrt[n]{n|a_n|}$$

$$= \lim_{n\to\infty} \sqrt[n]{n} \times \limsup \sqrt[n]{|a_n|}.$$

$$= 1 \times \frac{1}{R}.$$

$\Rightarrow R = R_d$. ∎

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Cor: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$, then f is diff. on $(-R,R)$

& $f'(x) = \underbrace{\sum_{n=1}^{\infty} n a_n x^{n-1}}_{\text{The derived P.S.}}$ $\forall x \in (-R, R)$.

[$\therefore$ $\frac{d}{dx}$(P.S) = derived P.S. on the same region of conv.]

Proof: $\because$ The derived P.S. is u.c. on ~~[a,b] in~~ all closed intervals contained in $(-R, R)$, it follows that ~~f(x)~~ $f'(x)$ exists $\forall x \in (-R, R)$ &

$f'(x)$ = the derived sum.

using derivatives of series of u.c. fn's.

Cor: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. Then f has derivatives of all orders on $(-R, R)$.

Moreover:
$$\boxed{a_n = \frac{f^{(n)}(0)}{n!} \quad \forall \quad n \geq 0}.$$
#

Proof: $f^{(n)}(x)$ exists $\forall n \geq 0$ & $x \in (-R, R)$ follows from the previous Corollary.

The equality follows from induction.

Remark: The above corollary $\Longrightarrow$ P.S. is !.

i.e. if $f : (-R,R) \longrightarrow \mathbb{R}$ is a fn. s.t. $f(x) = \sum_{n=0}^{\infty} a_n x^n$,

& if $f(x) = \sum_{n=0}^{\infty} b_n x^n$ on $(-R,R)$, then

$a_n = b_n \quad \forall n.$

$\boxtimes$

Remark: Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$. (Assume $R > 0$).

Then we already proved:

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \forall\ n \geq 0.$$

$$\therefore\ f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

on $(-R, R)$

The Taylor series of the fn. f about 0.

Treat this as the defn. of Taylor series.

Q: ~~Let~~ Let f be a fn. that is infinitely diff. ~~at 0~~ in a nbd of 0, say on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

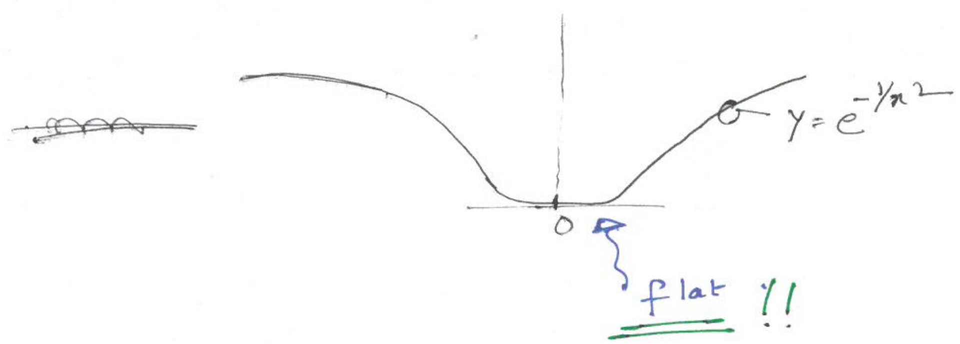
$$\overset{?}{\Longrightarrow} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad \forall\ x \in (-\varepsilon, \varepsilon)?$$

The Taylor series of f around 0.

or $f(x) = \begin{cases} e^{-1/x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Ans: No !!

eg: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$



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Then f is infinitely diff. ~~at one~~ on $\mathbb{R}$.

Easy to see $\forall x \in \mathbb{R}\setminus\{0\}$.

At $x=0$: Check (HW).

Moreover: $f^{(n)}(0) = 0 \quad \forall \; n \geq 0$.

$\Rightarrow$ The Taylor expansion of f around 0 is:

$$\sum \frac{f^{(n)}(0)}{n!} x^n \equiv 0 .$$

$$\Rightarrow \quad f(x) \neq \underbrace{\sum \frac{f^{(n)}(0)}{n!} x^n}_{\equiv 0} \qquad \forall x \in (-\varepsilon, \varepsilon) \text{ for any } \varepsilon > 0 \,!!$$

\# You will face this in Complex analysis !!

Def: Let $f : (a,b) \to \mathbb{R}$ be a fn. We say that f is analytic at $c \in (a,b)$ if there is a P.S. about c that represents f in a nbd of c. i.e:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \qquad \forall \; x \in (c-\delta, c+\delta) \text{ for some } \delta > 0 .$$

\# f is analytic at c $\iff$ $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n, \; \forall x \in (c-\delta, c+\delta)$.

(The Taylor series of f about c.)

Remark: Clearly, if f is analytic at c, then f is smooth at c [i.e. $f^{(n)}(c)$ exists $\forall\, n \geq 0$].

Of course, smooth $\not\Longrightarrow$ analytic. "Complex analysis"

eg: $f(x) = \frac{1}{1-x}$ is analytic at $c \in \mathbb{R} \setminus \{1\}$.

Infact: remember: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1$.

In general: if $c \neq 1$, then:

$$f(x) = \frac{1}{1-c}\left[\frac{1}{1 - \frac{x-c}{1-c}}\right]$$

$$= \frac{1}{1-c} \times \sum_{n=0}^{\infty}\left(\frac{x-c}{1-c}\right)^n \quad \forall \left|\frac{x-c}{1-c}\right| < 1.$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-c)^{n+1}} (x-c)^n. \quad \forall\, x \in \mathbb{R} \text{ s.t. } |x-a| < |1-c|.$$

$\therefore$ f is defined on all of $\mathbb{R} \setminus \{1\}$, but, the above equality holds only on $|x-a| < |1-c|$.

$\therefore$ The radius of convergence of the above P.S. is $|1-c|$ & interval of conv. is

$$(a - |1-c|,\ a + |1-c|).$$

All depends on c.