Def: Let $\mathcal{O} \subseteq \mathbb{R}$ be open, & let $f : \mathcal{O} \to \mathbb{R}$. We say that f is analytic on $\mathcal{O}$ if it is analytic at each $c \in \mathcal{O}$. i.e. $\forall\, c \in \mathcal{O}$, $\exists\, \delta\ (\equiv \delta(c)) > 0$ & $\{a_n\} \subseteq \mathbb{R}$ (again: $a_n = a_n(c)$) s.t.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \qquad \forall\, x \in \mathcal{O} \cap (\delta - c, \delta + c).$$

[$\Leftrightarrow$ f admits P.S./Taylor exp. about c $\forall\, c \in \mathcal{O}$].

Note: Often we say "Real anlytic" instead of analytic: But ~~this~~ can wait till Complex analysis.

real analytic

eg: Let $\sum_{n=0}^{\infty} a_n x^n$ be a P.S. with radius of convergence $R > 0$.

Then $f(x) := \sum_{n=0}^{\infty} a_n x^n$ is analytic on $(-R, R)$.

WHY?

Ans:

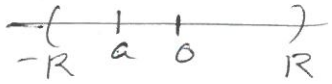
Thm: Let $\sum_{n=0}^{\infty} a_n x^n := f(x)$ has radius of convergence $R > 0$.

Suppose $a \in (-R, R)$. Then the Taylor series expansion of f about a is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n,$$

for all $x \in \mathbb{R}$ s.t. $|x - a| < R - |a|$.

<u>Proof:</u> Fix $a \in (-R, R)$.



Set $\delta := R - |a|$.

We use $x^n = ((x-a)+a)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} (x-a)^m$. $\forall\ n \geq 0$.

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} a^{n-m} (x-a)^m.$$

Set $\alpha_{m,n} := \begin{cases} \binom{n}{m} & \forall m = 0, 1, \dots, n \\ 0 & \text{oth. (i.e. } m > n) \end{cases}$ $\cancel{\sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} \binom{n}{m} a_n a^{n-m} \right) (x-a)^m}$.

WAIT

"double series."

$$\therefore \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \alpha_{m,n} a^{n-m} (x-a)^m. \quad \text{--- } (\dagger)$$

Note that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left| a_n \alpha_{m,n} a^{n-m} (x-a)^m \right|$ ← Testing A.C of double series

$$= \sum_{n=0}^{\infty} |a_n| \sum_{m=0}^{\infty} \binom{n}{m} |a|^{n-m} |x-a|^m.$$

$$= \sum_{n=0}^{\infty} |a_n| \left(|x-a| + |a| \right)^n. \quad \text{--- } (*)$$

If $|x-a| < \delta := R - |a| \Rightarrow |x-a| + |a| < R$.

$\Rightarrow (*)$ Converges.

$\Rightarrow$ The double series in $(\dagger)$ is absolutely "Convergent". ← [Convergence of double series will be discussed later.]

[A double series $\sum a_{m,n}$ is <u>A.C</u>. if $\sum_{m,n} |a_{m,n}|$ converges

"Due"

$$\Longleftrightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty \Leftrightarrow \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{m,n}| < \infty .$$

AND: In this case: ← "due"/"pending fact":

$$\left[\sum_{m,n} a_{m,n} = \sum_{m} \sum_{n} a_{m,n} = \sum_{n} \sum_{m} a_{m,n} \right].$$

Change of order of summations

$\therefore$ By changing the order of summation in (†):

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_n \alpha_{m,n} a^{n-m} (x-a)^m .$$

$\forall x$ s.t. $|x-a| < R-|a|$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} a_n \binom{n}{m} a^{n-m} \right) (x-a)^m .$$

Recall: $\alpha_{m,n} = \begin{cases} \binom{m}{n} & m \leq n \\ 0 & m > n. \end{cases}$

$$= \sum_{m=0}^{\infty} \tilde{a}_m (x-a)^m .$$

where $\tilde{a}_m := \sum_{n=m}^{\infty} a_m \binom{n}{m} a^{n-m} \quad \forall m \geq 0 .$

i.e. $$\sum_{n=0}^{\infty} a_n x^n = \sum_{m=0}^{\infty} \tilde{a}_m (x-a)^m$$

$\forall x$ s.t. $|x-a| < \delta = R-|a|.$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is analytic at a. ▨

Given a fn. $f : S \xrightarrow{\subseteq \mathbb{R},} \mathbb{R}$, denote by $Z(f)$ the zero set of f. i.e.

$$\therefore \quad Z(f) = \{x \in S : f(x) = 0\}.$$

If $p \in \mathbb{R}[x]$, then $\# Z(p) < \infty$.

Q: What about $Z\left(\sum_{n=0}^{\infty} a_n x^n\right)$? OR, Z (Analytic fn)

Ans: "Like" polynomials. ??

⟶ Next page ⟶

~~Thm: Let $f : O \to \mathbb{R}$ be an analytic fn., where O is an open interval.~~

~~Thm: Let $f : (a,b) \to \mathbb{R}$ be an analytic fn. If $Z(f)$ has a limit point in (a,b), then $f \equiv 0$.~~

~~Proof: Let c be a limit point of $Z(f)$ & $c \in (a,b)$.~~

~~$\Rightarrow f(c) = 0$. $[\because f$ is cont. at $c]$~~

~~$\Rightarrow Z(f)' \subseteq Z(f)$.~~

Iso $Z(f) = \vec{f}(0)$

~~If possible, let $f \not\equiv 0$.~~

~~We know: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ on $(c-\delta, c+\delta)$~~

~~$\because f \not\equiv 0, \exists$~~

PTO

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First, observe that if $f : (a,b) \longrightarrow \mathbb{R}$ is analytic, then f is Cont: on (a,b) $\quad [\because f$ is diff. on $(a,b)]$

$$\Rightarrow \mathcal{Z}(f) = f^{-1}(\{0\}) \text{ is a closed set.}$$

$$\therefore \mathcal{Z}(f)' \subseteq \mathcal{Z}(f)$$

Set of limit points of $\mathcal{Z}(f)$.

(a,b) is important. $\therefore (0,1) \cup (3,4)$ may not work!!

Thm: Let $f \not\equiv 0$ be an analytic fn on (a,b). Then $\mathcal{Z}(f)$ ~~does not~~ cannot have a limit point in (a,b).

($\Leftrightarrow$ $\mathcal{Z}(f)$ is a set of some isolated points)

Proof: We prove zeros of f are isolated.

Set $\mathcal{O} := \{x \in (a,b) : f^{(n)}(x) = 0 \quad \forall n = 0,1,\dots\}$.

$\therefore$ If $c \in \mathcal{O}$, then $f^{(n)}(c) = 0 \quad \forall n \geq 0$.

$\because f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ in a nbd of c,

it follows that $f \equiv 0$ in a nbd of c.

$\Rightarrow \mathcal{O}$ is open. (possibly φ).

Next, assume that $c \in (a,b) \setminus \mathcal{O}$.

$$\Rightarrow \exists\, m \geq 0 \text{ s.t. } f^{(m)}(c) \neq 0.$$

m may depend on c.

But $f^{(m)}$ is also analytic at c. ← Why?

$\because f^{(m)}(c) \neq 0$, by continuity of $f^{(m)}$ at c, it follows that $f^{(m)}(x) \neq 0$ in a nbd of ~~...~~ c contained in $(a,b) \setminus \mathcal{O}$.

Recall: $x_0 \in \mathcal{O} \Leftrightarrow f^{(n)}(x_0) = 0 \;\; \forall n \geq 0$

$\Rightarrow (a,b) \setminus \mathcal{O}$ is also open.

$\therefore$ Both $\mathcal{O}$ & $(a,b) \setminus \mathcal{O}$ are open.

But (a,b) is a connected set (or an interval).

$\Rightarrow$ either $\mathcal{O} = \varphi$ or $(a,b) \setminus \mathcal{O} = \varphi$.

$\Rightarrow$ either $\mathcal{O} = \varphi$ or $\mathcal{O} = (a,b)$. $\Leftrightarrow f \equiv 0$.

$\therefore$ If $f \not\equiv 0$, then zeros of f are isolated points.

Indeed

If $c \in Z(f)$, then by $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$,

we know that $\exists\, m \in \mathbb{N}$ s.t.

$$0 = f(c) = f^{(1)}(c) = \cdots = f^{(m-1)}(c) \;\; \&$$

$$f^{(m)}(c) \neq 0.$$

$$\Rightarrow f(x) = (x-c)^m \times \left(\text{"} \frac{f^{(m)}(c)}{m!} + \frac{f^{(m+1)}(c)}{(m+1)!}(x-c) + \cdots \text{"} \right)$$

$$= (x-c)^m \times g(x)$$

defined in a nbd of c.

" " $= g$: A P.S. with same radius of convergence.

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But g is also analytic at c & $g(c) \neq 0$. $\leftarrow$ ($\because f^{(m)}(c) \neq 0$.)

$\therefore$ By cont. $g(x) \neq 0$ $\forall x$ in a nbd of c.

$\Rightarrow f(x) \neq 0$ $\forall x \in$ deleted nbd of c.

$\Rightarrow c$ is an isolated point. ∎

"Identity thm".

Cor: Let $f, g : (a,b) \longrightarrow \mathbb{R}$ analytic. If $f(z) = g(z)$ $\forall z \in A$ s.t. $A' \cap (a,b) \neq \varphi$, then $f = g$ on (a,b).

— x —

Thm: (Abel's thm (1826))

Let $\sum_{n=1}^{\infty} a_n$ converges. Then the series $\sum_{n=0}^{\infty} a_n x^n$ ~~converges~~ is uniformly conv. on $[0,1]$.

Works for $R > 0$.

$\because \sum a_n$ conv. it follows that $\sum_{n=0}^{\infty} a_n x^n$ conv. on $(-1, 1]$ & A.C. on $(-1,1)$.

$\Leftarrow$

Enough to prove that:

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n .$$

As: If $f(x) := \sum_{n=0}^{\infty} a_n x^n$, then f is a p.s. with radius of conv ≥ 1.

The above $\Rightarrow \lim_{x \to 1^-} f(x) = f(1) \Rightarrow f$ is cont. on $[0,1]$.

<u>Proof:</u> Set $f(x) := \sum_{n=0}^{\infty} a_n x^n$. $|x| < 1$.

(We know rad. of conv. ≥ 1.)

<u>Claim:</u> $\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$.

Set $\alpha_n := \sum_{k=0}^{n-1} a_k$. We know $\alpha_n \to \sum a_n := \alpha$.

Also set $S_n(x) := \sum_{k=0}^{n-1} a_k x^k$. ← n-th partial sum of $\sum a_n x^n$.

By Abel's lemma: $$S_n(x) = \sum_{k=0}^{n-1} \alpha_k (x^k - x^{k+1}) + \alpha_n x^n.$$

$$\Rightarrow S_n(x) = \sum_{k=0}^{n-1} \alpha_k (1-x) x^k + \alpha_n x^n.$$

$\therefore \forall x \in (0,1)$, as $n \to \infty$, we have:

$$f(x) = (1-x) \times \sum_{n=0}^{\infty} \alpha_n x^n$$

$[\because \alpha_n x^n \to 0]$ as $\alpha_n \to \sum a_n$ & $x^n \to 0$.

$$\Rightarrow f(x) - \underbrace{\sum_{n=0}^{\infty} a_n}_{= (1-x) \times \sum_{n=0}^{\infty} x^n \times \sum_{n=0}^{\infty} a_n} = (1-x) \times \sum_{n=0}^{\infty} (\alpha_n - \alpha) x^n$$

$\alpha := \sum_{n=0}^{\infty} a_n$

$\forall\ 0 < x < 1$.

i.e. $$f(x) - \alpha = (1-x) \sum_{n=0}^{\infty} (\alpha_n - \alpha) x^n. \quad \forall\ 0 < x < 1.$$

Let $\varepsilon > 0$. As $\alpha_n \to \alpha$, $\exists\ N \in \mathbb{N}$ s.t.

$$|\alpha_n - \alpha| < \varepsilon/2 \quad \forall\ n \geq N.$$

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$$\therefore \ |f(x)-\alpha| \leq |1-x| \times \sum_{n=0}^{\infty} |\alpha_n - \alpha| x^n.$$

$\forall \ 0<x<1$

$$= |1-x| \times \left\{ \sum_{n=0}^{N-1} |\alpha_n-\alpha| x^n + \sum_{n=N}^{\infty} |\alpha_n - \alpha| x^n \right\}.$$

$|1-x| = 1-x$

$$< |1-x| \times \left\{ \sum_{n=0}^{N-1} |\alpha_n - \alpha| + \frac{\varepsilon}{2} \times \sum_{n=N}^{\infty} \cancel{|\alpha_n-\alpha|} x^n \right\}$$

$\because x<1$

$$< (1-x) \times \left\{ \sum_{n=0}^{N-1} |\alpha_n - \alpha| + \frac{\varepsilon}{2} \times (1-x)^{-1} \right\}.$$

$\because 0<x<1$

$$= (1-x) \times \sum_{n=0}^{N-1} |\alpha_n - \alpha| + \varepsilon/2.$$

~~Now $x \to 1^- \Rightarrow (1-x)$~~

Now for the same $\varepsilon > 0$, $\exists \ \delta > 0$ s.t.

$$1-x < \frac{\varepsilon}{2 \times \sum_{n=0}^{N-1} |\alpha_n - \alpha|} \quad \forall \ 0 < 1-x < \delta.$$

$$\therefore \ |f(x) - \alpha| < \varepsilon \quad \forall \ 0 < 1-x < \delta.$$

$$\Rightarrow \lim_{x \to 1^-} f(x) = \alpha \quad \left(= \sum_{n=0}^{\infty} a_n \right)$$

$\boxed{Q}$

Similar technique with the help of Cauchy Criterion for uniform convergence $\Rightarrow$.

Thm: (Abel's thm) If $\sum a_n x^n$ conv. on $(-R, R)$ & if $\sum_{n=0}^{\infty} a_n R^n$ Converges, then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly converges on $[0, R]$.

Recall: $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$, $|x| < 1$.

By integ. term-by-term

$$\Rightarrow \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\forall \; x \in (-1, 1).$$

i.e. $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ on $(-1, 1)$. $\leftarrow$ We already know this.

$\because$ $1 - \frac{1}{2} + \frac{1}{3} - \dots$ Conv. by Abel's thm.

$$\boxed{\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots}$$

This is an exciting equality!!

i.e. Alt-harmonic series $= \ln 2$!!