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Double Sequences

Def: A ~~real valued~~ fn. $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ $\left(\text{or } f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R} \right)$ is called a double seqn.

We write f simply as $\{f(m,n)\}$ or $\underbrace{\{a_{m,n}\}_{m,n \in \mathbb{N}}}_{a_{m,n} := f(m,n) \ \forall (m,n) \in \mathbb{N} \times \mathbb{N}}$.

eg: $\left\{ \frac{1}{m+n} \right\}_{m,n \in \mathbb{N}}$, $\left\{ e^{mn} \right\}_{m,n \in \mathbb{N}}$, $\left\{ m + \cos mn \right\}_{m,n \in \mathbb{Z}_+}$

Obs: For each fixed $m \in \mathbb{N}$, $\{a_{m,n}\}_{n=1}^{\infty}$ is a seqn.

||ly ——— $n \in \mathbb{N}$, $\{a_{m,n}\}_{m=1}^{\infty}$ —"— .

$\therefore$ It make sense to talk about:

$$\lim_{n \to \infty} \left(\lim_{m \to \infty} a_{mn} \right) \ \& \ \lim_{m \to \infty} \left(\lim_{n \to \infty} a_{mn} \right).$$

Q: $\underset{\cdots}{\downarrow} \overset{??}{=} \underset{\cdots}{\downarrow}$

Ans: No.

Alt:

$$\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & \cdots & \longrightarrow & a_1 \ (\text{say}) \\
a_{21} & a_{22} & a_{23} & \cdots & \longrightarrow & a_2 \ (\text{say}) \\
\vdots & \vdots & \vdots & & \vdots & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & \longrightarrow & a_m \ (\text{say}) \\
\vdots & \vdots & \vdots & & & \downarrow m \to \infty \\
\downarrow & \downarrow & \downarrow & & \xrightarrow[m \to \infty]{} & \text{same limit ??} \\
b_1 & b_2 & b_3 & & &
\end{array}$$

eg: $a_{m,n} = \dfrac{n}{m+n} \quad \forall\, m, n \geq 1.$

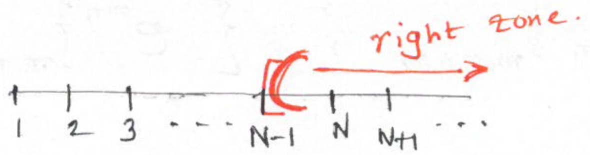
$$\therefore \lim_{n \to \infty} a_{m,n} = 1 \neq 0 = \lim_{m \to \infty} a_{m,n}.$$

$$\therefore \lim_{m \to \infty}\left(\lim_{n \to \infty} a_{m,n}\right) \neq \lim_{n \to \infty}\left(\lim_{m \to \infty} a_{m,n}\right) \text{ in general.}$$

Q: How to define convergency of $\{a_{m,n}\}$?

For $\{a_n\}$, we say $a_n \to a$ if for $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon \quad \forall\, n \geq N$.

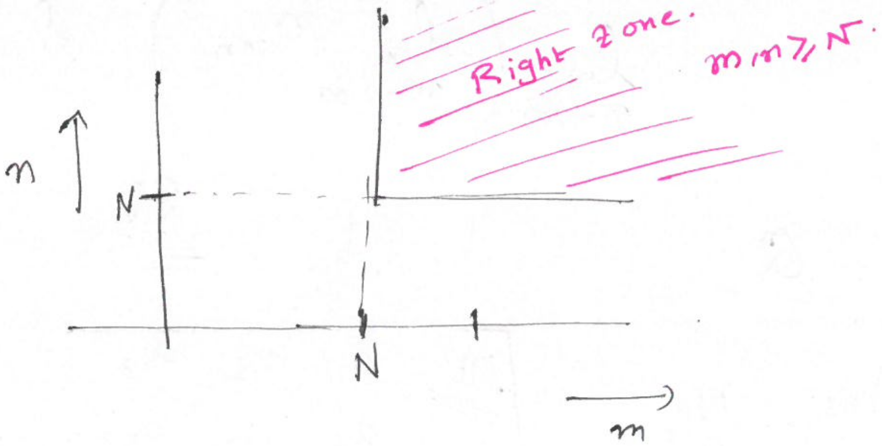
i.e.



Similarly

Def: A double seqn. $\{a_{m,n}\}$ converges to the double limit a if for $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t.

$$|a_{m,n} - a| < \varepsilon \quad \forall\, m, n \geq N.$$



Def: If $\{a_{m,n}\}$ does not converge, we say that it diverges.

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Def: Iterated limits of the double seq^n $\{a_{m,n}\}$ are:

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{m,n}\right) \quad \& \quad \lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right).$$

Thm: Let $a_{m,n} \longrightarrow a$ as $m,n \to \infty$. If $\lim_{n\to\infty} a_{m,n}$ exists $\forall m$, then $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right) = a$.

[Illy if $\lim_{m\to\infty} a_{m,n}$ exists $\forall n$, then $\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{m,n}\right) = a$]

Proof: Set $\alpha_m := \lim_{n\to\infty} a_{m,n}$ $\forall m$. [Claim: $\alpha_m \to a$].

Fix $\varepsilon > 0$. $\exists\, N \in \mathbb{N}$ s.t.

$$|a_{m,n} - a| < \varepsilon/2 \quad \forall\, m,n \geq N$$

$\because$ $\lim_{n\to\infty} a_{m,n} = \alpha_m$ for all m, for each $m \in \mathbb{N}$

$\exists$ $N(m) \in \mathbb{N}$ s.t.

$$|a_{m,n} - \alpha_m| < \varepsilon/2 \quad \forall\, n \geq N(m).$$

Fix $m \geq N$. Then pick $n \in \mathbb{N}$ s.t. $n \geq N(m)$ & $n \geq N$.

$$\therefore\ |\alpha_m - a| \leq |\alpha_m - a_{m,n}| + |a_{m,n} - a|$$

$$< \varepsilon/2 + \varepsilon/2 .$$

$$\Rightarrow |\alpha_m - a| < \varepsilon \quad \forall\, m \geq N.$$

$$\Rightarrow \alpha_m \longrightarrow a.$$

$\boxtimes$

eg: ~~" $\Leftarrow$ " i.e. $\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right)$ exists $\not\Rightarrow$ $\lim_{m,n\to\infty} a_{m,n}$ exists.~~

\# $a_{m,n} := (-1)^{m+n}\left(\frac{1}{m}+\frac{1}{n}\right)$. $\Rightarrow$ $|a_{m,n}| = \left|\frac{1}{m}+\frac{1}{n}\right|$ $\Rightarrow$ $a_{m,n} \to 0$ as $m,n\to\infty$

However, for each $m \in \mathbb{N}$, $\{a_{m,n}\}$ div. & also for each $n \in \mathbb{N}$, $\{a_{m,n}\}$ div. $\Rightarrow$ iterated limit DNE !!

Thm: (Cauchy criterion)

$\{a_{m,n}\}$ Converges $\Leftrightarrow$ for $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t.

$$|a_{m,n} - a_{p,q}| < \varepsilon \quad \forall\ m \geq p \geq N \ \ \& \ \ n \geq q \geq N. \qquad \text{—— } (*)$$

Proof: " $\Rightarrow$ " Let $a_{m,n} \to a$ as $m,n \to \infty$.

Let $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ s.t.

$$|a_{m,n} - a| < \varepsilon/2 \quad \forall\ m, n \geq N.$$

$\therefore$ For $m \geq p \geq N$ & $n \geq q \geq N$,

$$|a_{m,n} - a_{p,q}| \leq |a_{m,n} - a| + |a_{p,q} - a|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

" $\Leftarrow$ " Let $\varepsilon > 0$. Then $\exists$ $N \in \mathbb{N}$ s.t. $(*)$ holds.

$\forall n \in \mathbb{N}$, set $\alpha_n := a_{n,n}$. $\leftarrow$ "the diagonal."

$\therefore (*) \Rightarrow$ $|\alpha_n - \alpha_p| < \varepsilon \quad \forall\ n \geq p \geq N$

$\Rightarrow$ $\{\alpha_n\} \subseteq \mathbb{R}$ is Cauchy.

$\Rightarrow$ $\{\alpha_n\}$ Converges. Let $\alpha_n \to a$ as $n \to \infty$.

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$\therefore$ For $\varepsilon > 0$ $\exists$ $N_0 \in \mathbb{N}$ s.t.

$$|\alpha_n - a| < \varepsilon/2 \quad \forall\, n \geq N_0$$

Set $\tilde{N} := \max\{N, N_0\}$.

"Clever trick"

$\therefore$ $\forall\, m, n \geq \tilde{N}$, we have:

$$|a_{m,n} - a| \leq |a_{m,n} - \underbrace{a_{n,n}}_{\alpha_n}| + |\underbrace{a_{n,n}}_{\alpha_n} - a|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\Rightarrow a_{m,n} \longrightarrow a.$ $\boxed{//}$

— x —

§ Double Series:

Given a double seq^{n} $\{a_{m,n}\}$, we set

$$S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \quad \forall\ m, n \geq 1.$$

(m,n)-th partial sum.

The double seq^{n} $\{S_{m,n}\}$ is said to be the double series generated by $\{a_{m,n}\}$, & denoted by $\sum_{m,n=1}^{\infty} a_{m,n}$.

If $\lim_{m,n\to\infty} S_{m,n} = a$ (i.e. converges),

then we say that $\sum_{m,n=1}^{\infty} a_{m,n}$ converges & write

$$\sum_{m,n=1}^{\infty} a_{m,n} = a.$$

HW: Let $\sum_{m,n=1}^{\infty} a_{m,n}$ converges. Then $a_{m,n} \to 0$ as $m,n \to \infty$.

~~eg~~ HW: Let $a_{m,n} \geq 0 \ \forall m,n$. Then $\sum_{m,n=1}^{\infty} a_{m,n}$ converges $\iff$ the double seqn. $\{S_{m,n}\}_{m,n \geq 1}$ is bounded.

eg: ~~Let $p,q > 1$. Then for each~~
Let $\alpha, \beta > 1$. Then $\sum_{m,n=1}^{\infty} \frac{1}{\alpha^m \beta^n}$ is Convergent.

Proof. $$S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{\alpha^i \beta^j}.$$

$$< \left(\sum_{i=1}^{m} \frac{1}{\alpha^i} \right) \times \left(\sum_{j=1}^{n} \frac{1}{\beta^j} \right).$$

$\because$ $\sum_{n=1}^{\infty} \frac{1}{\alpha^n}$ & $\sum_{n=1}^{\infty} \frac{1}{\beta^n}$ are Convergent, $\alpha, \beta > 1$.

$\left\{ \sum_{i=1}^{m} \frac{1}{\alpha^i} \right\}_{m \geq 1}$ is a bdd seqn.

IIIy $\left\{ \sum_{i=1}^{n} \frac{1}{\beta^i} \right\}_{n \geq 1}$ is a bdd seqn.

$\Rightarrow$ $\{S_{m,n}\}_{m,n \geq 1}$ is a bdd seqn.

$\Rightarrow$ $\sum_{m,n=1}^{\infty} 1/\alpha^m \beta^n$ Converges.

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Comparison test:

Let $\sum a_{m,n}$ & $\sum b_{m,n}$ be two double series. Suppose $a_{m,n}, b_{m,n} \geq 0 \ \forall\, m,n$ & also let

$$a_{m,n} \leq b_{m,n} \quad \forall\, m,n.$$

If $\sum b_{m,n}$ conv. then $\sum a_{m,n}$ conv. — HW-.

The pending result-

you will encounter this in measure theory.

Thm:- (Fubini - Tonelli theorem for series).

A double series $\sum\limits_{m,n=1}^{\infty} a_{m,n}$ is absolutely convergent

$\Longleftrightarrow$ one (& hence, both) of the following conditions hold:

(i) $\sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} |a_{m,n}| < \infty$,

(ii) $\sum\limits_{n=1}^{\infty} \sum\limits_{m=1}^{\infty} |a_{m,n}| < \infty$.

Moreover, in this case: $\sum\limits_{m,n=1}^{\infty} a_{m,n} = \sum\limits_{m=1}^{\infty} \sum\limits_{n=1}^{\infty} a_{m,n} = \sum\limits_{n=1}^{\infty} \sum\limits_{m=1}^{\infty} a_{m,n}$.

Proof:- As usual, set $S_{m,n} = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} a_{ij}$

& $\gamma_{m,n} = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} |a_{ij}|$. $\forall\, m,n \geq 1$

"$\Leftarrow$" is now obvious! ← OR, wait till the end.

"$\Rightarrow$" Let $\sum_{m,n=1}^{\infty} |a_{m,n}| < \infty$.

Let $\varepsilon > 0$.
By Cauchy criterion: $\exists\, N \in \mathbb{N}$ s.t.

$$|r_{m,n} - r_{p,q}| < \varepsilon \quad \forall \begin{rcases} m \geq p \geq N \\ n \geq q \geq N \end{rcases}$$

Now $|s_{m,n} - s_{p,q}| \leq |r_{m,n} - r_{p,q}| < \varepsilon \quad \forall$ — " —

$\therefore$ By Cauchy criterion, again, $\sum a_{m,n}$ converges.

Set: $a := \sum_{m,n=1}^{\infty} a_{m,n}$.

Also, set $r := \sup_{m,n} r_{m,n}$ $\quad$ $\left[\sum |a_{mn}| < \infty \text{ \& } r_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| \right)$ $\Downarrow$ $r_{m,n} \uparrow$.

$\therefore \forall \begin{rcases} i \in \mathbb{N} \\ \& \, n \in \mathbb{N} \end{rcases}$, $\sum_{j=1}^{n} |a_{i,j}| \leq r_{i,n} \leq r$.

$\Rightarrow \forall i \in \mathbb{N}$, $\sum_{n=1}^{\infty} |a_{i,n}| < \infty \Rightarrow \sum_{n=1}^{\infty} a_{i,n}$ converges $\forall i \in \mathbb{N}$.

$\forall m \in \mathbb{N}$, Set $\alpha_m := \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{i,j}$.

$\because$ $a = \sum_{m,n=1}^{\infty} a_{m,n}$, for $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t.

$$|s_{m,n} - a| < \varepsilon \quad \forall m, n \geq N.$$

i.e. $\left| \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} - a \right| < \varepsilon \quad \forall m, n \geq N$.

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Fix m & let $n \to \infty$. $\Longrightarrow$

$$|\alpha_m - a| \leq \varepsilon \quad \forall\ m \geq N.$$

$$\Longrightarrow \alpha_m \longrightarrow a \quad \text{as } m \to \infty.$$

$$\left.\begin{array}{l} \therefore \displaystyle\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n}. \\ \text{Illy} \displaystyle\sum_{m,n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}. \end{array}\right\}$$

Finally (the pending ~~case~~ issue):

Let $\alpha := \displaystyle\sum_{m,n=1}^{\infty} |a_{m,n}|$. $\qquad r_{m,n} = \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|$.

$$\therefore r_{m,n} \leq \alpha \quad \forall\ m, n \geq 1.$$

Also, for each $m \in \mathbb{N}$, the seqn. $\{r_{m,n}\} \uparrow$.

$$\Rightarrow \lim_{n \to \infty} r_{m,n} \leq \alpha. \quad \forall\ m \in \mathbb{N}.$$

Also, observe that $r_{m,n} \leq r_{p,q} \quad \forall \left\{\begin{array}{l} m \leq p \\ \& n \leq q \end{array}\right\}$.

$$\therefore \lim_{m \to \infty} \lim_{n \to \infty} r_{m,n} < \infty$$

$$\text{i.e } \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty. \qquad \boxed{\text{H}}$$