$\therefore$ We have:

$$L(f,P) \leqslant S(f,P) \leqslant U(f,P). \qquad \forall P \in \mathcal{P}[a,b].$$

BUT, this depends on ~~tag~~ tag sets.!!

We predict: $f \in \mathcal{R}[a,b] \iff$ "$\exists\ \lambda \in \mathbb{R}$ s.t. $S(f,P) \longrightarrow \lambda$ as $\|P\| \longrightarrow 0$."

Meaning?

Whatever it is, ~~that~~ we will be in a good situation, as λ doesn't depends on tag set!!

~~Meaning~~

Def: Given $f \in \mathcal{B}[a,b]$, we say that

$$\lim_{\|P\| \to 0} S(f,P) = \lambda, \quad \text{for some } \lambda \in \mathbb{R}, \text{ if}$$

for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$|S(f,P) - \lambda| < \varepsilon \qquad \forall P \in \mathcal{P}[a,b] \ \& \ T_P \text{ (tag set) s.t. } \|P\| < \delta.$$

Danger: "$\forall T_P$" is a part of the definition.

Fact: The limit is !.

Proof: Suppose $\lim_{\|P\| \to 0} S(f,P) = \lambda_1$ & $\lim_{\|P\| \to 0} S(f,P) = \lambda_2$, for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

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If not, let $|\lambda_1 - \lambda_2| := \varepsilon > 0$.

$\therefore \exists\ \delta_1, \delta_2 > 0$ s.t.

$$|S(f,P) - \lambda_1| < \varepsilon/2 \qquad \forall\ \|P\| < \delta_1$$

$$\& \quad |S(f,P) - \lambda_2| < \varepsilon/2 \qquad \forall\ \|P\| < \delta_2$$

$\therefore$ For $\delta := \min\{\delta_1, \delta_2\}$, we have:

As usual, $\forall$ tags too.

$$|S(f,P) - \lambda_1| < \varepsilon/2 \ \& \ |S(f,P) - \lambda_2| < \varepsilon/2 \qquad \forall\ \|P\| < \delta.$$

$$\begin{aligned} \therefore \quad \varepsilon = |\lambda_1 - \lambda_2| &\leq |S(f,P) - \lambda_1| + |S(f,P) - \lambda_2| \quad \forall\ \|P\| < \delta. \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \qquad \rightarrow\leftarrow. \end{aligned}$$

$\Rightarrow \lambda_1 = \lambda_2$. $\boxtimes$

And, the good!! :

Thm: Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff \exists\ \lambda \in \mathbb{R} \ni$

$$\lim_{\|P\| \to 0} S(f,P) = \lambda.$$

In this case, $\int_a^b f = \lambda$.

Note: (1) Again, "$S(f,P) \to \lambda \quad \forall\ T_P$".

(2) "$\Rightarrow$" part is more remarkable & effective.

i.e. Suppose we know that $f \in \mathcal{R}[a,b]$.

Alright: Consider $f \in C[a,b]$. ← As for example.

Consider $\underline{P}: a = x_0 < x_0+h < x_0+2h < \cdots < x_0+(n-1)h < x_0+nh = b$.

Consider the tag set $\underline{T_P}$ as: $\{x_0+jh\}_{j=0}^{n-1}$ OR $\{x_0+jh\}_{j=1}^{n}$

& compute $S(f,P)$. ← No seqn. in the statement. But we will get in to it soon.

And, in this case, $\int_a^b f =$ "$\lim_{n\to\infty} S(f,P)$" is simply the

limit of "Newton sums"!! ← "∴ School integ. is justified!!"

Proof: "⟹" Let $f \in \mathcal{R}[a,b]$. Suppose $\lambda := \int_a^b f$.

Fix $\varepsilon > 0$.

$\because f \in \mathcal{R}[a,b]$, $\exists\ \delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \quad \forall P \in \mathcal{P}[a,b] \text{ s.t. } \|P\| < \delta. \qquad \text{—— } (*)$$

Darboux Criterion.

We know that

$$L(f,P) \leq \underset{\text{\& } \forall T_P}{S(f,P)} \leq U(f,P) \quad \text{—— } (+)$$

$$\forall P \in \mathcal{P}[a,b]$$

$(*) \Rightarrow$ $\forall \|P\| < \delta$,

Now $\underline{U(f,P)} < \varepsilon + L(f,P) \leq \varepsilon + \underline{\int} f = \underline{\varepsilon + \lambda}$;

& $\underline{L(f,P)} > U(f,P) - \varepsilon \geq \overline{\int} f - \varepsilon = \underline{\lambda - \varepsilon}$;

$\therefore (+) \Rightarrow \quad \lambda - \varepsilon < S(f,P) < \lambda + \varepsilon \quad \forall \|P\| < \delta$ & T_P.

$$\Rightarrow |S(f,P) - \lambda| < \varepsilon \quad \text{—— } \| \text{ ——}.$$

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$$\Rightarrow \lim_{\|P\| \to 0} S(f, P) = \lambda.$$

"$\Leftarrow$" Suppose $\lambda := \lim_{\|P\| \to 0} S(f, P)$ exists.

Let $\varepsilon > 0$. $\therefore \exists\ \delta > 0$ s.t.

$$|S(f, P) - \lambda| < \frac{\varepsilon}{3} \quad \forall\ \|P\| < \delta \ \& \ T_P.$$

$$\Rightarrow \quad \lambda - \varepsilon/3 < S(f, P) < \lambda + \frac{\varepsilon}{3} \quad \text{—}\|\text{—}. \quad \text{——} (\star)$$

Recall: $S(f, P) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1})$; where $\{\xi_j\} = T_P$ AND $\xi_j \in [x_{j-1}, x_j]$.

(i) P is fixed. T_P is NOT.

(ii) A finite sum.

(iii) f is bounded.

$\Downarrow$

By taking inf ($\&$ sup) over $\xi_j \in [x_{j-1}, x_j]$

[i.e. inf $\&$ sup over T_P]

by $(\star)$, we have:

$$\left.\begin{array}{l} \lambda - \varepsilon/3 \leqslant L(f, P) \leqslant \lambda + \varepsilon/3 \\ \& \quad \lambda - \varepsilon/3 \leqslant U(f, P) \leqslant \lambda + \varepsilon/3 \end{array}\right\} \text{——} (\ddagger)$$

$$\therefore \quad U(f, P) - L(f, P) \leqslant \lambda + \varepsilon/3 - (\lambda - \varepsilon/3) = 2\varepsilon/3$$

$$\Rightarrow \quad \mathcal{U}(f,P) - L(f,P) < \varepsilon.$$

$\therefore$ Cauchy Criterion $\Longrightarrow$ $f \in \mathcal{R}[a,b]$.

Finally, $(\dagger) \Rightarrow$

$$\lambda - \varepsilon/3 \leqslant L(f,P) \leqslant \int_a^b f \leqslant \mathcal{U}(f,P) \leqslant \lambda + \varepsilon/3.$$

$$\Longrightarrow \quad \left| \lambda - \int_a^b f \right| < \varepsilon/3 \qquad \forall\, \varepsilon > 0 \text{ small}.$$

$$\Longrightarrow \quad \int_a^b f = \lambda. \qquad \boxtimes$$

Useful tool.

Thm: Suppose $f \in \mathcal{R}[a,b]$ & $\{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\|P_n\| \longrightarrow 0$.

Then $\lim_{n\to\infty} S(f, P_n) = \int_a^b f$.

$\forall$ Tag set T_{P_n}. i.e. limit is regardless of the choice of tag sets.

Proof: Let $\varepsilon > 0$. By Darboux criterion, $\exists\, \delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \qquad \forall\, \|P\| < \delta,\ P \in \mathcal{P}[a,b].$$

$\because \|P_n\| \longrightarrow 0$, $\exists\, N_0 \in \mathbb{N}$ s.t.

$$\|P_n\| < \delta \qquad \forall\, n \geqslant N_0.$$

ce $\quad \therefore \quad U(f,P_n) - L(f,P_n) < \varepsilon \qquad \forall\, n \geqslant N_0.$

$$\Longrightarrow \left[U(f,P_n) - \overline{\int_a^b} f \right] + \left[\underline{\int_a^b} f - L(f,P_n) \right] < \varepsilon.$$

$\because \int f = \overline{\int} f = \underline{\int} f$

$\because [\ -\|-\ ]$ & $[\ -\|-\ ] \geqslant 0$, it follows that:

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$$\left.\begin{array}{l} 0 \leq U(f, P_n) - \int_a^b f < \varepsilon \\ \& \ 0 \leq \int_a^b f - L(f, P_n) < \varepsilon. \end{array}\right\} \ \forall \ n \geq N_0.$$

$$\Rightarrow \left.\begin{array}{l} \lim_{n\to\infty} U(f, P_n) = \int_a^b f \\ \& \ \lim_{n\to\infty} L(f, P_n) = \int_a^b f. \end{array}\right\}$$

Remark: "..." part is a general fact !! $\because f \in \mathcal{R}[a,b] \overset{\circledast}{\Rightarrow} \exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$ $\left(= \int_a^b f\right.$.

$\circledast$ "$\Leftarrow$" is also true !!

Finally, since $U(f, P_n) \leq S(f, P_n) \leq L(f, P_n) \ \forall n$, by the Squeeze theorem:

$$\lim_{n\to\infty} S(f, P_n) = \int_a^b f.$$

Again, regardless of ~~tag~~ tags!!

▨

\# The above result is fair & very useful !!

$\boxed{\text{"School integration" verified \& justified.}}$

Consider $f \in C[a,b]$ $\longleftarrow$ "A School fn"

For $n \in \mathbb{N}$, consider $P_n : a = x_0 < x_1 < \cdots < x_n = b$ with

$$x_j - x_{j-1} = \frac{b-a}{n}. \quad \longleftarrow \text{"School partition".}$$

$$\therefore \|P_n\| = \frac{b-a}{n} \quad \forall \ n \in \mathbb{N}.$$

$$\Rightarrow \|P_n\| \longrightarrow 0.$$

Then for any tag set $\{\xi_j\}_{j=1}^{n}$, we have:

$$\boxed{\int_a^b f = \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{j=1}^{n} f(\xi_j)\right].} \quad \text{———} \ \circledast$$

" The School time $\xi_j := a + \frac{b-a}{n}(j-1)$ $\forall j = 1, \dots, n$.

end points

or $\xi_j := a + \frac{b-a}{n} j$ "

$\Downarrow$

The precise " School integration " !!

Remark: Of Course $\circledast$ holds for all $f \in \mathcal{R}[a,b]$!!

∴ So far, we have the following (Summary):

Let $f \in \mathcal{B}[a,b]$. TFAE:

① $f \in \mathcal{R}[a,b]$.

② [Cauchy Criterion]: For $\varepsilon > 0$ $\exists$ $P_\varepsilon \in \mathcal{P}[a,b]$ s.t.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

③ [Darboux Criterion]: For $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon \quad \forall P \in \mathcal{P}[a,b] \text{ with } \|P\| < \delta.$$

④ $\exists$ $\{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\lim_{n\to\infty}\left[U(f,P_n) - L(f,P_n)\right] = 0$.

[in this case: $\lim U(f,P_n) = \lim L(f,P_n) = \int_a^b f$.].

(5) $\exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t. $\lim\limits_{n\to\infty} L(f, P_n) = \lim\limits_{n\to\infty} U(f, P_n)$.

both exist.

(6) $\lim\limits_{\|P\|\to 0} S(f, P) = \lambda$ exists ($\forall$ tag set).

In this case: $\lambda = \int_a^b f$.

$\boxed{///}$

Remark: Recall: $\mathcal{R}[a,b] := \{ f \in \mathcal{B}[a,b] : f \text{ is integrable.} \}$

Also recall: # $\mathcal{B}[a,b]$ is a vector space.

$\forall f, g \in \mathcal{B}[a,b]$ & $r \in \mathbb{R}$,

$f + rg \in \mathcal{B}[a,b]$.

Here $(f + rg)(t) = f(t) + r g(t)$ $\forall t \in [a,b]$.

"Algebra" # Also $\forall f, g \in \mathcal{B}[a,b]$,

$fg \in \mathcal{B}[a,b]$. ← Why?

Here $(fg)(x) = f(x) g(x)$ $\forall x \in [a,b]$

good with compositions. # Also, if $[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}$

are bounded, then $g \circ f \in \mathcal{B}[a,b]$!!

Why?

$\therefore$ We can ask all the questions for $\mathcal{R}[a,b]$.
[by replacing $\mathcal{B}[a,b]$].

$\therefore$ $\mathcal{R}[a,b]$ a vector space? An algebra?

SECONDLY:

Consider ~~Suppose~~ $\mathcal{I} : \mathcal{R}[a,b] \longrightarrow \mathbb{R}$ defined by

$$\mathcal{I}(f) = \int_a^b f \qquad \forall\ f \in \mathcal{R}[a,b].$$

We need to think about "$\mathcal{I}$", $\mathcal{R}[a,b]$ & the Structure of $\mathcal{R}[a,b]$ together. Like:

"Linear".

$$\mathcal{I}(f \pm rg) = \mathcal{I}(f) + r\,\mathcal{I}(g) \ ?$$

This is really a natural question.

"multiplicative"

$$\mathcal{I}(fg) = \mathcal{I}(f)\,\mathcal{I}(g) \ ?$$

well, no harm in asking!!

"order preserving".

If $f \leq g$ (i.e. $f(x) \leq g(x)\ \forall\ x \in [a,b]$), then $\mathcal{I}(f) \leq \mathcal{I}(g)$?

splits?

Let $a < c < b$, & $f \in \mathcal{R}[a,b]$.

$$\overset{?}{\Longrightarrow} \quad \mathcal{I}(f) = \int_a^c f + \int_c^b f \ ?$$

ETC.!!