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Thm: Let $f, g \in \mathcal{R}[a,b]$ & $r \in \mathbb{R}$. Then
$f + rg \in \mathcal{R}[a,b]$ & $\boxed{\int_a^b f + rg = \int_a^b f + r\int_a^b g}$.

[In particular: $\int f \pm g = \int f \pm \int g$; $\int cf = c\int f$ etc.]

Note: Presenting one out of many proofs.

Proof: Note that $(f + rg)(x) = f(x) + r\,g(x) \;\forall\, x \in [a,b]$.

First, we consider rf.

Recall: For $P \in \mathcal{P}[a,b]$ & T_P a tag set,

$$S(f,P) = \sum_{j=1}^{n} f(\xi_j)\,|I_j| \quad ; \quad \begin{array}{l} P := a = x_1 < \cdots < x_n = b. \\ \xi_j \in I_j := [x_j, x_{j-1}] \\ \quad \forall j = 1, \dots, n \\ \& \; T_P = \{\xi_j\}_{j=1}^{n}. \end{array}$$

$\therefore\; S(rf, P) = r\,S(f,P)$. $\longleftarrow$ $(\because (rf)(x) = rf(x) \;\forall\, x.)$.

$$\Rightarrow \left| S(rf,P) - r\int_a^b f \right| = |r| \left| S(f,P) - \int_a^b f \right| \quad \forall\, P \in \mathcal{P}[a,b]$$

$\because f \in \mathcal{R}[a,b]$, the RHS can be made smallest.

$$\Rightarrow \underline{rf \in \mathcal{R}[a,b]} \;\; \& \;\; \underline{\int_a^b rf = r\int_a^b f}.$$

$\therefore$ It is now enough to prove that $\underline{\int_a^b f+g = \int_a^b f + \int_a^b g}$.

But again: $S(f+g, P) = S(f,P) + S(g,P)$

$$\therefore \left| S(f+g,P) - \int_a^b f - \int_a^b g \right| = \left| \left(S(f,P) - \int_a^b f\right) + \left(S(g,P) - \int_a^b g\right) \right|$$

$$\leqslant \left| S(f,P) - \int_a^b f \right| + \left| S(g,P) - \int_a^b g \right|$$

$\because f, g \in \mathcal{R}[a,b]$, for $\varepsilon > 0$ $\exists\, \delta_1, \delta_2 > 0$ s.t.

$$\left| S(f,P) - \int_a^b f \right| < \varepsilon/2 \quad \forall\, \|P\| < \delta_1$$

&

$\&$ $\left| S(g,P) - \int_a^b g \right| < \varepsilon/2 \quad \forall \, \|P\| < \delta_2$.

Choose $0 < \delta < \min\{\delta_1, \delta_2\}$.

$\therefore \left| S(f+g, P) - \int_a^b f - \int_a^b g \right| < \varepsilon/2 + \varepsilon/2 \quad \forall \, \|P\| < \delta$ $\& \, T_p$.

$\Rightarrow f+g \in \mathcal{R}[a,b]$ $\&$ $\int_a^b f+g = \int_a^b f + \int_a^b g$.

$\boxed{/}$

A ~~Very~~ very Serious H.W:

Let $B \subseteq \mathbb{R}$ be a bdd set. Then $\widehat{B} := \{|x-y| : x, y \in B\}$ is also bdd. Moreover:

$$\operatorname{Sup} \widehat{B} = \operatorname{Sup} B - \inf B.$$

$$\Downarrow$$

Fact: Let $f \in \mathcal{B}[a,b]$. Then $\forall \, P \in \mathcal{P}[a,b]$, we have

$$M - m = \operatorname{Sup}\{|f(x) - f(y)| : x, y \in [a,b]\}$$

$$\& \quad M_j - m_j = \operatorname{Sup}\{|f(x) - f(y)| : x, y \in [x_{j-1}, x_j]\}.$$

$$\forall \, j = 1, \cdots, n.$$

useful.

$\&$ where: P: $a = x_0 < \cdots < x_n = b$.

Often, $\operatorname{osc}_{I_i} f := \operatorname{Sup}\{|f(x) - f(y)| : x, y \in I_i\}$.

The oscillation of f on I_i.

$i = 1, \cdots, n$.

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In general:

Def: For $A \subseteq \mathbb{R}$ & a bdd fn $f: A \to \mathbb{R}$, the oscillation of f is defined by:

$$\operatorname{osc}_A f := \sup_A f - \inf_A f .$$

$$\boxed{\therefore \operatorname{osc}_{I_j} f = M_j - m_j \ \forall j}$$ # ⟵ useful. & In general:

Fact: For $f \in \mathcal{B}[a,b]$ & $P \in \mathcal{P}[a,b]$, with $P: a = x_0 < x_1 < \cdots < x_n = b$,

we have $U(f,P) - L(f,P) = \sum_{j=1}^{n} (M_j - m_j)\, |I_j|$.

$$\Rightarrow \boxed{U(f,P) - L(f,P) = \sum_{j=1}^{n} \operatorname{osc}_{I_j} f \ |I_j|}$$

More arithmetic & properties of $\mathcal{R}[a,b]$:

Suppose $f, g \in \mathcal{R}[a,b]$. Then:

(1) $f + rg \in \mathcal{R}[a,b] \ \forall r \in \mathbb{R}$ & $\int_a^b f + rg = \int_a^b f + r\int_a^b g$.

— done. —

(2) $f^2 \in \mathcal{R}[a,b]$. (But $\int_a^b f^2 \neq \left(\int_a^b f\right)^2$: in general. — why — ??)

Proof: Clearly, $|f(x)|^2 \leq M^2 \ \forall x$

$\Rightarrow f^2 \in \mathcal{B}[a,b]$.

Let $\varepsilon > 0$.

By Cauchy Criterion, $\exists P_\varepsilon \in \mathcal{P}[a,b]$ s.t.

$$0 \leq U(f,P) - L(f,P) < \frac{\varepsilon}{2M}. \quad \text{———} \circledast$$

Now $|f(x)^2 - f(y)^2| = |f(x)+f(y)|\,|f(x)-f(y)|$

$\leq 2M\,|f(x)-f(y)|. \quad \forall\, x,y \in [a,b].$

$$\Rightarrow \underbrace{\sup_{x,y\in I_j} |f(x)^2 - f(y)^2|}_{\parallel \; \sup_{I_j} f^2 - \inf_{I_j} f^2} \leq 2M \underbrace{\sup_{x,y\in I_j} |f(x)-f(y)|}_{\parallel \; M_j - m_j}. \quad \forall\, j = 1,\cdots,n.$$

i.e.: $\underbrace{\sup_{I_j} f^2 - \inf_{I_j} f^2}_{=\underset{I_j}{\mathrm{osc}}\, f^2} \leq 2M.\underbrace{(M_j - m_j)}_{=\underset{I_j}{\mathrm{osc}}\, f} \quad \forall j = 1,\cdots,n.$

$$\therefore\ U(f^2,P) - L(f^2,P) = \sum_{j=1}^{n} \underset{I_j}{\mathrm{osc}}\, f \times |I_j|$$

$$\leq 2M \times (U(f,P) - L(f,P))$$

$$\underset{\text{by } \circledast}{<} 2M \times \frac{\varepsilon}{2M}.$$

$$\Rightarrow U(f^2,P) - L(f^2,P) < \varepsilon.$$

$$\Rightarrow \underline{f^2 \in \mathcal{R}[a,b]}. \qquad ///$$

(3) $\boxed{fg \in \mathcal{R}[a,b]}$. [Recall: $(fg)(x) = f(x)\,g(x)$]

Proof: $\because fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$.

We have (1) & (2) $\Rightarrow fg \in \mathcal{R}[a,b]$.

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④ $|f| \in \mathcal{R}[a,b]$ & $\boxed{\left|\int_a^b f\right| \leq \int_a^b |f|}$

(Like: $\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|$!!)

Proof: We know $|p| - |q| \leq |p-q| \quad \forall\, p, q \in \mathbb{R}$.

$\therefore \left| |f(x)| - |f(y)| \right| \leq |f(x) - f(y)| \quad \forall\, x, y \in [a,b]$. ——— ⊛

$\Rightarrow$ Let P be a partition of $[a,b]$. & $P: a = x_0 < \cdots < x_n = b$.

$\therefore$ ⊛ $\Rightarrow \sup_{I_j} |f| - \inf_{I_j} |f| \leq M_i - m_i$

$\Rightarrow (\quad -\text{"}- \quad) \times |I_j| \leq (-\text{"}-) \times |I_j|$.

$\Rightarrow U(|f|, P) - L(|f|, P) \leq U(f,P) - L(f,P)$.

$\because f \in \mathcal{R}[a,b]$, it follows that $|f| \in \mathcal{R}[a,b]$.

"$\left|\int f\right| \leq \int |f|$" follows from the next observation.

⑤ If $f \geq 0$ (i.e. $f(x) \geq 0 \;\; \forall\, x \in [a,b]$), then

$$\boxed{\int_a^b f \geq 0.}$$

Proof: $\because f \geq 0$, $\forall\, P \in \mathcal{P}[a,b]$,

$L(f, P) \geq 0$.

$\Rightarrow \underline{\int_a^b} f \geq 0.$ But $\underline{\int_a^b} f = \int_a^b f \Rightarrow \int_a^b f \geq 0.$

(6) If $f(x) \leq g(x)$ $\forall x \in [a,b]$ (i.e. $f \leq g$), ~~then~~ $\int_a^b f \leq \int_a^b g.$

Proof: $f \leq g \Rightarrow g - f \geq 0. \overset{\text{by (5)}}{\Rightarrow} \int (g-f) \geq 0.$

$\overset{\text{by (1)}}{\Rightarrow} \int g - \int f \geq 0 \Rightarrow \int g \geq \int f.$

(7) $\left| \int_a^b f \right| \leq \int_a^b |f| \quad \forall f \in \mathcal{R}[a,b].$

Proof: (4) $\Rightarrow |f| \in \mathcal{R}[a,b].$

Now $-|f| \leq f \leq |f|.$

Now, (6) $\Rightarrow -\int |f| \leq \int f \leq \int |f|$

$\Rightarrow \left| \int f \right| \leq \int |f|.$ ☒

(8) $\max\{f,g\}, \min\{f,g\} \in \mathcal{R}[a,b].$

$\left(\max\{f,g\}\right)(x) = \max\{f(x), g(x)\}. \quad \forall x \in [a,b].$

Proof: Use: $\max\{p,q\} = \dfrac{(p+q) + |p-q|}{2}$

& $\min\{p,q\} = \dfrac{p+q-|p-q|}{2}$

$\forall p,q \in \mathbb{R}.$

& all the previous observations.

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(9) Let $\frac{1}{g} \in \mathcal{B}[a,b]$. Then $\frac{f}{g} \in \mathcal{R}[a,b]$.

Proof: Enough to prove that $\frac{1}{g} \in \mathcal{R}[a,b]$.

$$\left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| = \frac{1}{|g(x)|\,|g(y)|} \times |g(x) - g(y)|$$

$$\leq \tilde{M}^2 \times |g(x) - g(y)|$$

where $\tilde{M} := \sup_{x \in [a,b]} \left| \frac{1}{g(x)} \right|$.

$\therefore$ For $P: a = x_0 < \cdots < x_n = b$ in $\mathcal{P}[a,b]$, we have:

$$\operatorname*{osc}_{I_j} \frac{1}{g} \leq \tilde{M}^2 \times \operatorname*{osc}_{I_j} g.$$

$$\Rightarrow \frac{1}{g} \in \mathcal{R}[a,b].$$

Feel free to adopt this for simpler proofs of earlier observations.

Observation (general): If $\operatorname*{osc}_{I_j} G \leq \operatorname*{osc}_{I_j} F$ $\forall j$ & $\forall P \in \mathcal{P}[a,b]$, then $F \in \mathcal{R}[a,b] \Rightarrow G \in \mathcal{R}[a,b]$.

DANGER

$G \leq F$ & $F \in \mathcal{R}[a,b]$ $\not\Rightarrow G \in \mathcal{R}[a,b]$!!

eg: ~ HW ~.

(10) Let $a < c < b$. Then $f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f|_{[c,b]} \in \mathcal{R}[c,b]$ &

$$\boxed{\int_a^b f = \int_a^c f + \int_c^b f}.$$

Proof: Set $f_1 = f|_{[a,c]}$ & $f_2 = f|_{[c,b]}$. Clearly, $f_1 \in \mathcal{B}[a,c]$ & $f_2 \in \mathcal{B}[c,b]$.

Let $\varepsilon > 0$. By Cauchy Criterion: $\exists\, P \in \mathcal{P}[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon.$$

WLOG: let $c \in P$ (i.e. c is a node). Otherwise replace the above P by $P \cup \{c\}$ & get the same inequality $[\because P \cup \{c\} \supset P]$.

Set: $P: \underbrace{a = x_1 < \cdots < x_{m-1} < x_m}_{P_1} = c \underbrace{< x_{m+1} < \cdots < x_n = b}_{P_2}$

Then $P_1 : a = x_1 < \cdots < x_m = c$ in $\mathcal{P}[a,c]$
& $P_2 : c = x_m < \cdots < x_n = b$ in $\mathcal{P}[c,b]$ $\Big\} \Rightarrow P = P_1 \cup P_2$.

$$\Rightarrow \underbrace{[U(f_1,P_1) - L(f_1,P_1)]}_{\geq 0} + \underbrace{[U(f_2,P_2) - L(f,P_2)]}_{\geq 0} = U(f,P) - L(f,P) < \varepsilon.$$

$$\Rightarrow\# \quad U(f_j, P_j) - L(f_j, P_j) < \varepsilon \quad \forall\, j = 1, 2. \quad \text{—— } (*)$$

$$\Rightarrow f_1 \in \mathcal{R}[a,c] \;\&\; f_2 \in \mathcal{R}[c,b].$$

Set $\lambda_1 := \int_a^c f_1$ & $\lambda_2 := \int_c^b f_2$ (Note: $\lambda_1 = \int_a^c f$ & $\lambda_2 = \int_c^b f$).

Claim: $\int_a^b f = \lambda_1 + \lambda_2$.

Indeed, ($P \in \mathcal{P}[a,b]$ as above)

$$\int_a^b f \geq L(f,P) = L(f_1,P_1) + L(f_2,P_2)$$
$$\underset{\text{by}(*)}{>} U(f_1,P_1) + U(f_2,P_2) - 2\varepsilon. \geq \lambda_1 + \lambda_2 - 2\varepsilon.$$

& Illy,

$$\int_a^b f \leq U(f,P) = U(f_1,P_1) + U(f_2,P_2)$$
$$\underset{\text{by}(*)}{<} L(f_1,P_1) + L(f_2,P_2) + 2\varepsilon.$$
$$\leq \lambda_1 + \lambda_2 + 2\varepsilon.$$

$$\therefore \lambda_1 + \lambda_2 - 2\varepsilon \leq \int_a^b f \leq \lambda_1 + \lambda_2 + 2\varepsilon. \quad \forall\, \varepsilon > 0.$$

$$\Rightarrow \int_a^b f = \lambda_1 + \lambda_2 \quad \text{i.e.} \quad \int_a^b f = \int_a^c f + \int_c^b f. \quad \square$$