Jaydeb Sarkar.

Fact: Let $f \in \mathcal{R}[a,b]$. ~~Then~~

① $m(b-a) \leq \int_a^b f \leq M(b-a)$. ← We know this.

② If ~~in addition~~, $f \in C[a,b]$, then $\exists\ c \in [a,b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

← Kind of MVT !!

Proof: $\because$ f is cont. on $[a,b]$ (← compact), f attains all the values in $[m, M]$.

Then $m \leq \frac{1}{b-a}\int_a^b f \leq M$

$\implies \exists\ c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$. ∎

Def: ① $\int_a^a f := 0$ $\quad \forall$ fn $f : S \longrightarrow \mathbb{R}$, $S \ni a$.

~~Fact~~ ② $\forall f \in \mathcal{B}[a,b]$,

$$\int_b^a f := -\int_a^b f.$$

Compositions

Thm: Let $f \in \mathcal{R}[a,b]$, ran $f \subseteq [c,d]$ & let $g \in C[a,b]$.

Then $g \circ f \in \mathcal{R}[a,b]$.

$$\begin{tikzcd} {[a,b]} \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & {[c,d]} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

[Recall: $(g \circ f)(x) = g(f(x))$ $\forall x \in [a,b]$.].

Proof: Clearly $g \circ f \in \mathcal{B}[a,b]$.

Let $\varepsilon > 0$. Since $g : [c,d] \longrightarrow \mathbb{R}$ is unif. cont.

$\exists\ \delta > 0$ s.t.

$$|g(x) - g(y)| < \frac{\varepsilon}{2(b-a)} \quad \forall\ |x-y| < \delta. \quad (*)$$

Set $M_1 := \sup\limits_{y \in [c,d]} |g(y)|$.

$\because f \in \mathcal{R}[a,b]$, by Cauchy criterion, $\exists\ P \in \mathcal{P}[a,b]$

s.t. $\qquad U(f,P) - L(f,P) < \dfrac{\varepsilon\delta}{4M_1}$ ——— (**)

Set $\quad P : a = x_0 < x_1 < \cdots < x_n = b$.

[Claim: $\quad U(g\circ f, P) - L(g \circ f, P) < \varepsilon$.]

Let $\quad J = \{1, \dots, n\}$. & write

$$J = J_1 \coprod J_2$$

disjoint partition.

where: $\quad J_1 = \{ j \in J : \quad M_j - m_j < \delta \}$

& $\quad J_2 = \{ j \in J : \quad M_j - m_j \geq \delta \}$.

Recall: $M_j - m_j = \operatorname{osc}\limits_{I_j} f$. $\quad \forall\ j = 1, \dots, n$.

$$\because \quad M_j - m_j = \operatorname{osc}_{I_j} f$$

$$= \sup_{x,y \in I_j} |f(x) - f(y)|$$

$$\forall\ j = 1, \dots, n,$$

We have: If $j \in J_1$, then $\quad |f(x) - f(y)| < \delta \quad \forall x, y \in I_j$

& by (*): $\quad |g(f(x)) - g(f(y))| < \dfrac{\varepsilon}{2(b-a)} \quad \forall x, y \in I_j$

$$\Longrightarrow \sup_{x,y \in I_j} |g(f(x)) - g(f(y))| \leq \frac{\varepsilon}{2(b-a)}.$$

For each $i = 1, \cdots, n$, we set

$$\tilde{M}_i := \sup_{I_i} g \circ f \quad \& \quad \tilde{m}_i := \inf_{I_i} g \circ f.$$

$$\therefore \forall j \in J_1, \quad \underbrace{\tilde{M}_j - \tilde{m}_j}_{= \operatorname{osc}_{I_j} g \circ f} \leq \frac{\varepsilon}{2(b-a)}.$$

$$\Rightarrow \sum_{j \in J_1} (\tilde{M}_j - \tilde{m}_j) \times |I_j| \leq \frac{\varepsilon}{2(b-a)} \times \sum_{j \in J_1} |I_j|.$$

$$\leq \frac{\varepsilon}{2(b-a)} \times (b-a).$$

$$\leq \varepsilon/2.$$

All about J_1

Next, we turn to J_2:

We note that $\forall j \in J_2$, $\tilde{M}_j - \tilde{m}_j \leq 2M_1$ $\quad M_1 = \sup_{[a,b]} |g|$

$$\therefore \sum_{j \in J_2} (\tilde{M}_j - \tilde{m}_j) \times |I_j| \leq 2M_1 \times \sum_{j \in J_2} |I_j|$$

$$\leq 2M_1 \times \sum_{j \in J_2} |I_j| \times \underbrace{\frac{(M_j - m_j)}{\delta}}_{\geq 1}.$$

$$[\because \forall j \in J_2,\ M_j - m_j \geq \delta]$$

$$= 2M_1 \times \frac{1}{\delta} \times \sum_{j \in J_2} |I_j| (M_j - m_j)$$

$$\leq 2M_1 \times \frac{1}{\delta} \sum_{j \in J} (M_j - m_j) |I_j|$$

$\because J_2 \subseteq J$

$$= \frac{2M_1}{\delta} \times (U(f,P) - L(f,P))$$

$$< \frac{2M_1}{\delta} \times \frac{\varepsilon \delta}{4M_1} \quad (\text{by } \ast\ast)$$

$$= \frac{\varepsilon}{2}.$$

$$\therefore \quad \mathcal{U}(g\circ f, P) - L(g\circ f, P) = \sum_{j\in J} (\tilde{M}_j - \tilde{m}_j)\, |I_j|$$

$$= \sum_{j\in J_1} + \sum_{j \in J_2} \quad \longleftarrow \quad \because J = J_1 \amalg J_2 .$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

$$\Rightarrow \quad g\circ f \in \mathcal{R}[a,b] .$$

$\boxtimes$

Cor: Suppose $f \in \mathcal{R}[a,b]$. Then:

① e^f, $\sin f(x)$, $\cos f(x)$ are in $\mathcal{R}[a,b]$.

② $\sqrt[n]{f} \in \mathcal{R}[a,b]$ $\forall n \geq 1$ whenever $f(x) \geq 0$.

Q: We proved: $f \in \mathcal{R}$, $g \in C \Rightarrow g\circ f \in \mathcal{R}$.

What about: "$f \in \mathcal{R}$ & $g \in \mathcal{R} \Rightarrow g \circ f \in \mathcal{R}$"?

False. Why ??

Thm: Suppose $f, g \in \mathcal{B}[a,b]$ & ~~for~~ $f(x) = g(x)$ $\forall x \in [a,b]$ but finitely many. Then $f \in \mathcal{R}[a,b]$ $\Leftrightarrow$ $g \in \mathcal{R}[a,b]$.

Moreover, in this case, $\displaystyle\int_a^b f = \int_a^b g$.

[$\because$ $f = 0$ on $[a,b]$ except finitely many points

$\Longrightarrow \displaystyle\int_a^b f = 0$.]

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Proof: Enough to assume

$$f(x) = g(x) \quad \forall \; x \in [a,b] \setminus \{c\}$$

& $f(c) \neq g(c)$.

for some $c \in [a,b]$.

The general case will then follow by induction.

So, assume the above conditions.

Let $\varepsilon > 0$ & suppose $\sup_{[a,b]} |f|, \sup_{[a,b]} |g| \leqslant \tilde{M}$.

for some $\tilde{M} > 0$.

[Claim: $\overline{\int} f = \overline{\int} g$ & $\underline{\int} f = \underline{\int} g$.].

For $\varepsilon > 0$ $\exists$ $P \in \mathcal{P}[a,b]$ s.t. $U(f,P) < \overline{\int} f + \varepsilon/2$. —— (*)

Set $\delta := \dfrac{\varepsilon}{8\tilde{M}}$.

Consider a refinement $\tilde{P}$ of P s.t. $\|\tilde{P}\| < \delta$

Always possible.

$\because$ $f = g$ on $[a,b]$ except $c \in [a,b]$,

f differs from g on atmost 2 subintervals of $\tilde{P}$.

[depending on c being an end point of an subinterval].

Let $\{\tilde{I}_j\}_{j=1}^n$ be the subintervals of $\tilde{P}$.

Assume f differs from g on I_l.

Here $\underbrace{l = p}_{\#}$, OR $\underbrace{l = p, p+1}_{\#}$, for some p.

$\Rightarrow$ $\sup_{I_j} f - \sup_{I_j} g = 0 \quad \forall \; j \neq p, p+1$

The max 2-point Case.

& $\left| \sup_{I_j} f - \sup_{I_j} g \right| \leqslant 2\tilde{M}$ for $j = p, p+1$.

$$\therefore \quad \left| U(f,\tilde{P}) - U(g,\tilde{P}) \right|$$

$$= \left| \left(\sup_{I_p} f - \sup_{I_p} g \right) \times |I_p| + \left(\sup_{I_{p+1}} f - \sup_{I_{p+1}} g \right) \times |I_{p+1}| \right|$$

$$\leqslant 2 \times 2\, \tilde{M} \times \delta .$$

i.e. $\left| U(f,\tilde{P}) - U(g,\tilde{P}) \right| \leqslant 4\delta\tilde{M} = 4\tilde{M} \times \frac{\varepsilon}{8\tilde{M}} = \varepsilon/2 .$

i.e. $\left| U(f,\tilde{P}) - U(g,\tilde{P}) \right| \leqslant \varepsilon/2 .$ ——— (**)

$\cancel{\text{So, } \overline{\int} g \leqslant U(g,\tilde{P}) \underset{\text{(by (*))}}{<} \overline{\int} f + \varepsilon/2}$

Thus, $\overline{\int} g \underset{\substack{\text{always} \\ \text{true.}}}{\leqslant} U(g,\tilde{P}) \underset{\text{by (**)}}{\leqslant} U(f,\tilde{P}) + \varepsilon/2$

$$\underset{\because\, \tilde{P} \supseteq P}{\leqslant} U(f,P) + \varepsilon/2 \underset{\text{by (*)}}{<} \left(\overline{\int} f + \varepsilon/2 \right) + \varepsilon/2$$

i.e. $\overline{\int} g < \overline{\int} f + \varepsilon \quad \forall \varepsilon .$

$\Rightarrow \overline{\int} g \leqslant \overline{\int} f .$

Switching f & g, this gives: $\overline{\int} f \leq \overline{\int} g .$

$\therefore \quad \overline{\int} f = \overline{\int} g .$

lly $\quad \underline{\int} f = \underline{\int} g .$

$\therefore \quad \overline{\int} f = \underline{\int} f \iff \overline{\int} g = \underline{\int} g .$ ///