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# §. The fundamental theorem of Calculus (FTC):

Note that if $\mathcal{D} :=$ diff. fn on $\mathbb{R}$ / $(a,b)$ / open set;

then $\quad d : \mathcal{D} \longrightarrow \mathbb{R}$ is a linear map.

$(df := f')$.

Illy. $\quad \mathcal{I} : \mathcal{R}[a,b] \longrightarrow \mathbb{R}$ is a linear map.

FTC ~~essen~~ essentially says: # "$d \circ \mathcal{I}$ = identity."

# "$\mathcal{I} \circ d$ = identity."

TROUBLE : Compositions should ~~be~~ be well-defined first !!

In the following, we will explain the informal equalities ~~&~~ ~~then~~ & then, ~~&~~ along with all the necessary assumptions, we will make it more formal !!

**Def:** Let $S \subseteq \mathbb{R}$ & $f : S \to \mathbb{R}$ be a fn. ~~An function~~

A differentiable fn $F$ is called an antiderivative or a primitive of $f$ on $S$ if

$$f(x) = F'(x) \quad \forall\, x \in S.$$

**eg:** i) $\frac{1}{2}x^2$ is an antiderivative of $x$.

ii) $\frac{1}{2}x^2 + c$ —— " —— $x$ $\forall\, c \in \mathbb{R}$.

$$\therefore \quad \underset{\text{antiderivative}}{F} \xrightarrow{\frac{d}{dx}} \underset{\text{function}}{f} \xrightarrow{\frac{d}{dx}} \underset{\text{derivative.}}{f'}$$

(PROVIDED: $f$ is diff.).

**Q:** Do all functions have antiderivatives ??

1) polynomials. ✓.

2) Continuous fn's. ✓ ← Why ?? ← WAIT (FTC-II).

3) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ ✗ ← Why ?

Ans.

"IVT"

**Digression** :

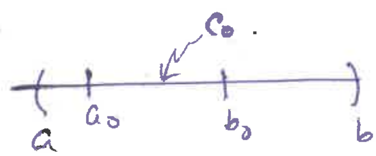
**Thm:** (Darboux's thm)

Let $f:(a,b) \longrightarrow \mathbb{R}$ be a diff. fn. & let $a < a_0 < b_0 < b$.

If $f'(a_0) < r < f'(b_0)$, then $\exists\ c_0 \in (a_0, b_0)$ s.t.

$$f'(c_0) = r.$$

impressive result.



**Proof:** [Note: If $f'$ is cont. then this is straight IVT !!].

Set $g(x) := f(x) - rx$. $x \in (a,b)$. (Here $r$ is fixed.)

$$\Rightarrow g \text{ diff } \& \begin{cases} g'(a_0) < 0 \\ g'(b_0) > 0 \end{cases}.$$

Also $g\big|_{[a_0,b_0]} : [a_0, b_0] \longrightarrow \mathbb{R}$ cont. & hence $g$ on $[a_0, b_0]$ is unif.

attain its extreme values.

Now $g'(a_0) > 0 \Rightarrow g(a_0+h) - g(a_0) > 0$ for $h>0$ small.

Why ??

$\Rightarrow g(a_0) < g(a_0+h)$ — " —.

$a_0 \quad a_0+h \qquad h>0$

$\Rightarrow g$ does not assumes the max. at $a_0$.

$|||^{ly}$ $g'(b_0) < 0 \Rightarrow g(b_0+h) - g(b_0) > 0$ for $h<0$ small.

Why ??

$\Rightarrow g(b_0) < g(b_0+h)$ for $h<0$ small.

$b_0+h \quad b_0 \qquad h<0.$

$\Rightarrow g$ does not assumes the max. at $b_0$.

$\therefore$ $g$ assumes ~~the~~ a max (max or local max) at $c_0 \in (a_0, b_0)$.

$\because$ $g$ is diff. $\quad g'(c_0) = 0$.

$\Rightarrow f'(c_0) = \gamma$. ▨

Note: Thus,

$$f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

does not have an antiderivative !!

~~Alert~~ ALERT: Derivatives need not be continuous !!

$$f(x) := \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

$f'$ is NOT Cont. at $0$.

THUS: Derivative of a fn need not be Cont. but still the derivative enjoys the Intermediate Value property!!

— the novelty of Darboux's thm.

Anyway;

The first FTC.

Thm: (FTC - I) Let $f \in \mathcal{R}[a,b]$ & $F \in C[a,b]$. Suppose $F$ is an antiderivative of $f$ over $(a,b)$. Then

$$\int_a^b f = F(b) - F(a).$$

① $\therefore F'(x) = f(x)$ $\forall x \in (a,b)$.

② Roughly: $\int_a^b F' = F(b) - F(a)$.

③

$$\int_{[a,b]} F' = [F]_{\partial[a,b]} = F(b) - F(a).$$

The solid interval.

boundary of $[a,b]$.

A remarkable result!!

* [In fact, we can redefine/assign $F'$ at $a$ & $b$. As $F'|_{(a,b)} = f$ & $f \in \mathcal{R}[a,b]$, the extended $F'$ on $[a,b]$ will be integrable & $\int_a^b (\text{extended } F') = \int_a^b F'$.]

④ A continuous analog of sums of differences!!

$$\text{"} \sum_{j=1}^{n} (\gamma_j - \gamma_{j-1}) = \gamma_n - \gamma_0 \text{."}$$

Proof: Let $P \in \mathcal{P}[a,b]$.

Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

$\therefore$ We have the sum of differences:

$$\sum_{j=1}^{n} \left( F(x_j) - F(x_{j-1}) \right) = F(b) - F(a).$$

Now $F\Big|_{[x_{j-1}, x_j]} \in C[x_{j-1}, x_j]$ & diff. on $(x_{j-1}, x_j)$.

$\forall j = 1, \dots, n.$

$\therefore$ MVT $\Rightarrow$ $\exists\, \xi_j \in (x_{j-1}, x_j)$ s.t.

$$F(x_j) - F(x_{j-1}) = F'(\xi_j)\,(x_j - x_{j-1}).$$

$$\Rightarrow F(x_j) - F(x_{j-1}) = f(\xi_j)\,(x_j - x_{j-1}) \quad \text{——} \quad \circledast.$$

as $F'(x) = f(x)$ $\forall x$ in $(a,b)$.

$\forall j = 1, \dots, n.$

Now $\forall j = 1, \dots, n$, we know: $\because \xi_j \in [x_{j-1}, x_j]$

$$m_j (x_j - x_{j-1}) \leq f(\xi_j)(x_j - x_{j-1}) \leq M_j (x_j - x_{j-1}).$$

$$\therefore \circledast \Rightarrow \quad m_j (x_j - x_{j-1}) \leq F(x_j) - F(x_{j-1}) \leq M_j (x_j - x_{j-1}) \quad \forall j.$$

$$\Rightarrow \quad L(f, P) \leq \sum_{j=1}^{n} \left( F(x_j) - F(x_{j-1}) \right) \leq U(f, P).$$

i.e. $L(f, P) \leq F(b) - F(a) \leq U(f, P) \quad \forall P \in \mathcal{P}[a,b].$

$$\Rightarrow \underline{\int} f \leq F(b) - F(a) \leq \overline{\int} f.$$

But $f \in \mathcal{R}[a,b]$. $\Longrightarrow$ $\displaystyle\int_a^b f = F(b) - F(a).$ ▨

$\therefore$ FTC-I $\Rightarrow$ $\int_a^b f$ can be computed by $\boxed{\text{finding}}$ antiderivative $F$ of $f$ !!

Q: How to find (of course, if any!!) an antiderivative ?

& we know, it may not be!!

— Ans: "FTC-II".

The 2nd FTC.

**Thm (FTC-II)**

Let $f \in \mathcal{R}[a,b]$. Define

$$F(x) := \int_a^x f(t)\,dt \qquad \forall\, x \in [a,b].$$

Then: ① $F \in C[a,b]$.

② If $f$ is cont. at $x_0 \in (a,b)$, then $F$ is diff. at $x_0$ & $F'(x_0) = f(x_0)$.

Remember? Integration makes fn smoother !!

③ If $f$ is cont. from the right at $a$, then $F_+'(a) = f(a)$. Illy cont. from left at $b$.

Proof: Set $M := \sup_{x \in [a,b]} |f(x)|$. Let $\varepsilon > 0$ & $x, y \in [a,b]$.

$$\therefore\ |F(x) - F(y)| = \left| \int_x^y f(t)\,dt \right|$$

$$\because\ -M \leq f(t) \leq M \qquad \forall\, t \in [a,b].$$

$$\Rightarrow -M(y-x) \leq (y-x) f(t) \leq (y-x) M \qquad \forall\, t \in [a,b].$$

[assuming $y > x$]

$\because$ the constant fn's $t \mapsto \pm M(y-x)$ are integrable

We have:

$$-M(\cancel{y-x}) \leq \int_x^y f(t)\,dt \leq M(\cancel{y-x}).$$

$$\Rightarrow \left| \int_x^y f(t)\,dt \right| \leq M(y-x). \qquad [\text{For } y > x].$$

$$\therefore \forall\ x, y \in [a,b], \qquad \left| \int_x^y f(t)\,dt \right| \leq M|x-y|.$$

i.e. $|F(x) - F(y)| \leq M|x-y|. \quad \forall\ x, y \in [a,b].$

$\Rightarrow$ F is uniformly ~~Cont~~. Cont. on $[a,b]$.

— This proves ①.

Now, for ②, let $f$ is Cont. at $x_0 \in (a,b)$.

$$\therefore \frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^{x} f(t)\,dt. \qquad \forall\ x \neq x_0.$$

Also, $$f(x_0) = \frac{1}{x - x_0} \int_{x_0}^{x} f(x_0)\,dt.$$

$$\therefore \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0} \int_{x_0}^{x} [f(t) - f(x_0)]\,dt. \qquad \forall\ x \neq x_0$$

Now for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $|f(t) - f(x_0)| < \varepsilon$ $\forall\ |t - x_0| < \delta$.

$$\text{Then, } \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \frac{1}{|x - x_0|} \left| \int_{x_0}^{x} [f(t) - f(x_0)]\,dt \right|$$

$$\leq \frac{1}{|x - x_0|} \times \int_{x_0}^{x} |f(t) - f(x_0)|\,dt.$$

$$< \frac{1}{|x-x_0|} \times \varepsilon \times |x-x_0| \qquad \forall\ |x-x_0|<\delta.$$

$$\Longrightarrow \left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| < \varepsilon \qquad \forall\ |x-x_0|<\delta.$$

$\Longrightarrow$ $F$ is diff. at $x_0$ & $F'(x_0) = f(x_0)$. $\blacksquare$

**Cor:** Let $f \in C[a,b]$. Then

$$\frac{d}{dx}\Bigg( \underbrace{\int_a^x f(t)\,dt}_{\text{differentiable.}} \Bigg) = f(x) \qquad \forall\ x \in [a,b].$$

$D \circ g$ = identity on $C[a,b]$.

# $\displaystyle \sum_{j=1}^{n} x_j - \sum_{j=1}^{n-1} x_j = x_n$ ⟵ analog of

difference of sums!!

**Fact:** Continuity of $f$ is a must for diff. of $x \mapsto \int_a^x f(t)\,dt$.

— HW. Will do it.

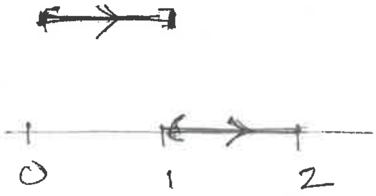
# In particular, if $f \in C[a,b]$, then

$x \longmapsto \int_a^x f(t)\,dt$ is an antiderivative of $f$.

And of course, we know $\exists\ f \in \mathcal{R}[a,b]$ ~~whi~~ with no antiderivatives !!

**eg:** $f : [0,2] \longrightarrow \mathbb{R}$ defined by :

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases}$$



Clearly, $f \in \mathcal{R}[0,2]$. $\Leftarrow$ $\because f = \chi_{[0,1]} \in \mathcal{R}[0,2]$.

Recall: if $A \subseteq B$, then

$\chi_A : B \longrightarrow \mathbb{R}$ defined by

indicator or characteristic fn.

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Set $F(x) := \int_0^x f(t)\,dt$. $\forall\, x \in [0,2]$.

If $x \in [0,1]$, then $F(x) = \int_0^x 1.dt = x$

If $x \in (1,2]$, then $F(x) = \int_0^1 f(t)\,dt + \int_1^x \underbrace{f(t)}_{=0}\,dt$.

$$= \int_0^1 1.dt + 0.$$

$$= 1 + 0 = 1.$$

$$\therefore F(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 0 & 1 < x \leq 2. \end{cases}$$

$\therefore$ F fails to be ~~cont.~~ diff. at $x = 1$ (precisely where $f$ is discont.).

$\therefore$ Integration of an integrable fn. need not be diff. !!

**Thm:** (Integration by parts):

Let $f, g \in \mathcal{D}[a,b]$ & $f', g' \in \mathcal{R}[a,b]$. Then

$$\int_a^b f g' + \int_a^b f' g = f(b)g(b) - f(a)g(a).$$

[ $f \in \mathcal{D}[a,b]$ means: $\exists$ diff. fn $F$ on $(a-\varepsilon, b+\varepsilon)$ s.t. $F|_{[a,b]} = f$.

OR $f$ is diff. on $(a,b)$ & $f'$ has an extension to $[a,b]$;

(So, extension to only 2 points : $a$ & $b$). ].

Negligible issue!!

**Proof:** Set $u = fg$. $\Rightarrow u' = f'g + fg'$.

FTC $\Rightarrow \int_a^b u' = u(b) - u(a)$.

$$\Rightarrow \int_a^b f g' + \int_a^b f' g = (fg)(b) - (fg)(a). \quad \square$$

[$\therefore \int_a^b f g' = [fg]_a^b - \int_a^b f' g$. $\Leftarrow$ the popular form!!]

**Thm:** (Change of variable): Let $I \subseteq \mathbb{R}$ be an open interval, $g : I \to \mathbb{R}$ diff. & $g' \in \mathcal{R}(C)$ $\forall$ closed interval $C \subseteq I$.

Set $J = g(I)$. $\leftarrow$ (also an interval as $g$ cont.)

If $f : J \to \mathbb{R}$ is cont. & $a < b$ in $I$, then

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(x)\, dx.}$$

**Proof:** $\because$ $f \circ g \in \mathcal{R}[a,b]$ (as $f$ cont), we have that $(f\circ g)\, g' \in \mathcal{R}[a,b]$

$\because$ $f \in C[g(a), g(b)]$, $F(x) := \int_{g(a)}^{x} f(t)\,dt$ is diff. & $F' = f$ on $[g(a), g(b)]$ ~~[a,b]~~

$[g(b), g(a)]$

Now $(F\circ g)'(x) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$ $\forall x \in [g(a), g(b)]$

$\Rightarrow F \circ g' \in \mathcal{R}[a,b]$.

$$\therefore \int_a^b f(g(x))\, g'(x)\, dx = \int_a^b (F\circ g)'(x)\, dx = (F\circ g)(b) - (F\circ g)(a) = F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} F'(x)\, dx. \quad \square$$

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## Change of variable

**Thm:** Let $u \in \mathcal{D}[a,b]$, $u' \in \mathcal{R}[a,b]$ & $f \in \mathcal{C}(u[a,b])$.

Then
$$\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f(x)\, dx.$$

The so called "u- substitution"

[ Note: $[a,b] \xrightarrow{u} u[a,b] \xrightarrow{f} \mathbb{R}$

$\Rightarrow f \circ u : [a,b] \longrightarrow \mathbb{R}$ ]

**Proof:** Note that $u$ = constant map $\Leftrightarrow u'(t) \equiv 0$.

Then the above equality is true (both sides $= 0$).

$\because u(a) = u(b)$.

So, assume that $u$ is non constant.

$\because f$ is cont, $f \circ u \in \mathcal{R}[a,b]$. As $u' \in \mathcal{R}[a,b]$, it follows that $(f \circ u)\, u' \in \mathcal{R}[a,b]$.

Also observe that $u[a,b]$ is an interval. Closed ?

$\forall x \in u[a,b]$, define $F(x) := \int_{u(a)}^{x} f(t)\, dt$.

By FTC-II, $F'(x) = f(x)$ $\forall x \in u[a,b]$.

Then ~~Also,~~ $(F \circ u)'(t) = F'(u(t))\, u'(t) = f(u(t))\, u'(t)$.

$\forall t \in [a,b]$

$\therefore$ FTC - I $\Rightarrow$

$$\int_a^b f(u(t))\, u'(t)\, dt = \int_a^b (F \circ u)'(t)\, dt$$

$$= (F \circ u)(b) - (F \circ u)(a)$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} F'(x)\, dx.$$

$$= \int_{u(a)}^{u(b)} f(x)\, dx. \qquad \boxed{//}$$

—— x ——.