Jaydeb Sarkar

Improper Integrals.

Assumptions for Riemann integrations:

$[a,b]$: i.e bdd & closed interval.

But we really want to integrate over (a,∞) or $(-\infty,a]$ or $(-\infty,\infty)$ etc.

$f \in \mathcal{B}[a,b]$ i.e. fn's must be bdd.

Even a fn is unbdd., there may be some chance of finite area !!

PRACTICAL ISSUES to deal with.

Improper integrals : $\int_a^b f$ where f is unbounded on $[a,b]$

$\int_a^\infty f$ or $\int_{-\infty}^b f$ or $\int_{-\infty}^\infty f$.

Goal: To find a way to make sense of improper integrals of above types !!

[Notation: $a, b \in \mathbb{R}$ & always: $a<b$].

Improper integrals of type I:

Let $f \notin \mathcal{R}[a,b]$ & Suppose $f \in \mathcal{R}[c,b]$ $\forall$ $a<c<b$. If $\lim_{c\to a^+} \int_c^b f$ exists, then we say that the I.I (improper integral) $\int_a^b f$ Converges & say

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f . \quad \left(\because \int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f \right). \qquad (A)$$

If the limit DNE, then we say $\int_a^b f$ diverges.

||ly let $f \notin \mathcal{R}[a,b]$ & $f \in \mathcal{R}[a,c]$ $\forall\ a<c<b$.

Then $\int_a^b f$ converges if

$$\lim_{c \to b^-} \int_a^c f \text{ exists. We write:}$$

$$\int_a^b f = \lim_{c \to b^-} \int_a^c f.$$

II of type I.

OR, Simply: $\boxed{\int_a^b f = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f.}$ —— (B)

Also ~~Finally~~: Let $a<c<b$. & suppose $\int_a^c f$ and/or $\int_c^b f$ are I.I. of type-I. (~~either~~ (A) or (B)).

We write $\int_a^b f := \int_a^c f + \int_c^b f$ —— (C)

if both I.I: $\int_a^c f$ & $\int_c^b f$ exist.

Otherwise, $\int_a^b f$ diverges.

||ly: If $a<c<b$ & f is unbounded ~~Bdosromtur~~ at $x=c$, then we write

$$\int_a^b f := \int_a^c f + \int_c^b f$$ provided both

I.I. in the RHS exist.

(B) Kind (A) Kind.

Jaydeb Sarkar.

eg:

(1) $\int_0^1 \frac{1}{x^2}\,dx.$

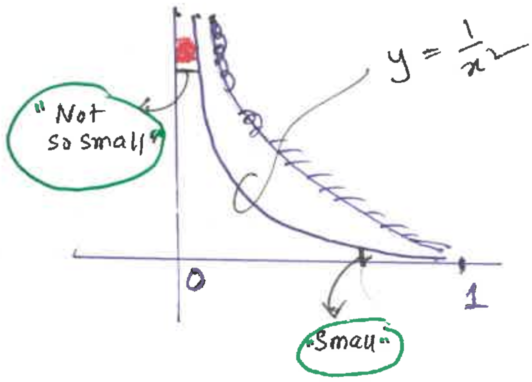
Clearly, this is an I.I. of type-I. $\leftarrow \because \frac{1}{x^2} \notin \mathcal{R}[0,1]$.

Now, $\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x^2}\,dx$ $\leftarrow \because \frac{1}{x^2} \in \mathcal{R}[\varepsilon,1]$ $\forall\, \varepsilon>0$ small.

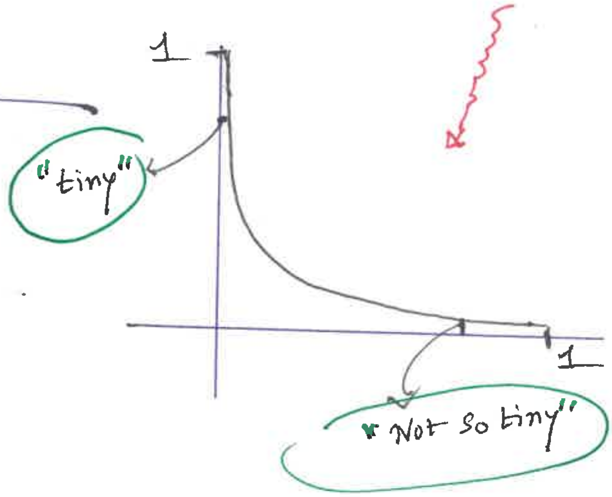
$= \lim_{\varepsilon\to 0^+} \left[-\frac{1}{x}\right]_\varepsilon^1$ $\leftarrow$ Why? FTC?

$= \lim_{\varepsilon\to 0^+} \left(\frac{1}{\varepsilon} - 1\right) = +\infty.$ $\leftarrow$ why?

$\therefore \int_0^1 \frac{1}{x^2}\,dx$ diverges.



(2) $\int_0^1 \frac{1}{\sqrt{x}}\,dx.$ $\leftarrow$ tells you we have a chance!!



Now $\forall\, \varepsilon>0$, $\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx$

$= 2\times\left[x^{1/2}\right]_\varepsilon^1 = 2(1-\sqrt{\varepsilon}).$

$\therefore \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+} 2(1-\sqrt{\varepsilon}) = 2.$

$\therefore \int_0^1 \frac{1}{\sqrt{x}}\,dx$ is Convergent &

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

③ $\int_0^2 \frac{1}{2x-x^2}\,dx$.

$x \xmapsto{f} \frac{1}{2x-x^2}$ is unbdd at $x = 0, 2$.

$\therefore$ We need to investigate

$$\int_0^1 f + \int_1^2 f.$$

or any $1 < c < 2$.

Now $\int_1^2 f = \lim_{\varepsilon\to 0^+} \int_1^{2-\varepsilon} \frac{1}{x(2-x)}\,dx.$

$\frac{1}{x(2-x)} = \left(\frac{1}{x} + \frac{1}{2-x}\right)\frac{1}{2}$

$$= \frac{1}{2}\lim_{\varepsilon\to 0^+} \int_1^{2-\varepsilon} \left(\frac{1}{x} + \frac{1}{2-x}\right) dx.$$

$$= \frac{1}{2}\lim_{\varepsilon\to 0^+} \left[\ln\left(\frac{x}{2-x}\right)\right]_1^{2-\varepsilon}$$

$$= \frac{1}{2}\lim_{\varepsilon\to 0^+} \ln\left(\frac{2-\varepsilon}{\varepsilon}\right) = \infty$$ ← why?

$\Rightarrow \int_0^2 \frac{1}{2x-x^2}\,dx$ diverges.

④ P.T. $\int_0^1 \frac{dx}{\sqrt{1-x}} = 2$. (HW).

Jaydeb Sarkar

In general:

Suppose $p > 0$.

Then $\displaystyle\int_0^1 \frac{dx}{x^p} = ??$ ⟵ Type I

Note that $\forall\ 0 < \varepsilon < 1$,

$$\int_\varepsilon^1 \frac{1}{x^p}\, dx = \begin{cases} \left[\dfrac{x^{-p+1}}{-p+1}\right]_\varepsilon^1 & \text{if } p \neq 1 \\ \left[\log x\right]_\varepsilon^1 & \text{if } p = 1. \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-p} \times \left(1 - \varepsilon^{1-p}\right) & \text{if } p \neq 1 \\ -\log \varepsilon & \text{if } p = 1 \end{cases}$$

$$\therefore \lim_{\varepsilon \to 0^+} \int \frac{1}{x^p}\, dx = \begin{cases} \dfrac{1}{1-p} & 0 < p < 1 \\ \infty & \text{if } p \geq 1. \end{cases}$$

$\therefore$ If $p > 0$, then the I.I. $\displaystyle\int_0^1 \frac{dx}{x^p}$ Converges for $0 < p < 1$ & diverges for $p \geq 1$.

[ZZ: $p < 0$ Case is not worth it!!
— why?]

~~Thm (Comparison test II)~~

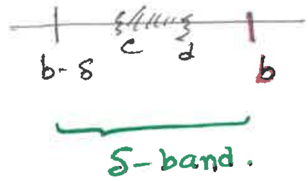
~~Result~~:

Natural classification.

Thm: Let $\int_a^b f$ be an I.I. at b. Then $\int_a^b f$ Converges $\iff$ for

$\varepsilon > 0 \ \exists \ \delta > 0$ s.t.

$$\left| \int_c^d f \right| < \varepsilon \qquad \forall \ c \ \& \ d \ \text{ s.t. } \ b-\delta < c < d < b.$$



Proof:

"$\Rightarrow$" Let $\int_a^b f$ Converges.

Set $F(x) := \int_a^x f(t)\,dt. \qquad x \in [a,b).$

Recall: $f \in R[a, b-\varepsilon]$ $\forall \ \varepsilon > 0$. ~~small~~.

By assumption: $\lim_{x \to b^-} F(x)$ exists.

$\therefore$ For $\varepsilon > 0$, $\exists \ \delta > 0$ s.t.

$$\underbrace{\left| F(c) - F(d) \right|}_{= \int_c^d f} < \varepsilon \qquad \forall \ c, d \ \text{ s.t. } \ b-\delta < c < d < b. \qquad \text{——— } (*)$$

$$\Rightarrow \left| \int_c^d f \right| < \varepsilon \qquad \text{— " —}.$$

"$\Leftarrow$" $(*)$ holds $\Rightarrow \lim_{x \to b^-} F(x)$ exists.

$$\Rightarrow \lim_{x \to b^-} \int_a^b f \ \text{ exists}. \qquad \boxed{\text{Q.E.D.}}$$

Thm: (Comparison test - I)

Let $0 \leq f(x) \leq g(x) \quad \forall \; x \in [a,b)$. Assume that $\int_a^b f$ & $\int_a^b g$ are I.I. at b.

① If $\int_a^b g$ converges, then $\int_a^b f$ converges.

② If $\int_a^b f$ diverges, then $\int_a^b g$ diverges.

Proof: ∵ ① ⇒ ②, we only prove ①.

Set $F(x) := \int_a^x g. \quad \forall \; x \in [a,b).$

∵ $g \geq 0$, $F \uparrow$. on $[a,b)$.

∵ $\int_a^b g$ converges, we have:

$$\int_a^b g = \lim_{x \to b^-} F(x)$$

$$= \sup \{ \underbrace{F(x)}_{= \int_a^x g(x)} : x \in [a,b) \}$$

$$= \sup \{ \int_a^x g(x) : x \in [a,b) \}.$$

Now $0 \leq f(x) \leq g(x) \quad \forall \; x \in [a,b)$

$$\Rightarrow 0 \leq \int_a^x f \leq \int_a^x g. \qquad \forall \; x \in [a,b).$$

$$\leq \int_a^b g.$$

$\therefore \quad x \longmapsto \int_a^x f \uparrow$ on $[a,b)$

& $\quad 0 \leq \int_a^x f \leq \int_a^x g \leq \int_a^b g.$

Then $\quad 0 \leq \int_a^x f \leq \int_a^b g$ & $x \longmapsto \int_a^x f \uparrow$

$\Rightarrow \lim_{x \to b^-} \int_a^x f$ exists & $\int_a^b f \leq \int_a^b g.$ $\boxed{\ }$

eg: For $p > 0$, Consider $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^p}\, dx.$

We know $x \longmapsto \dfrac{\sin x}{x}$ is bdd &

$$\frac{\sin x}{x} \leq 1 \qquad \forall\, 0 < x \leq \pi/2$$

$$\Rightarrow \frac{\sin x}{x^p} \leq \frac{1}{x^{p-1}} \qquad \text{— " —}$$

Now $\displaystyle\int_0^{\pi/2} \frac{1}{x^{p-1}}\, dx$ Converges only when $p-1<1$

$\Leftrightarrow p < 2.$

$\therefore$ By Comparison test:

$\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^p}\, dx$ Converges for $p<2$

& diverges for $p \geq 2$.

$\boxed{\ }$