Thm: (Limit Comparison test - I)

Let $f(x), g(x) \geq 0 \quad \forall\, x \in [a,b)$.

Suppose $\lim\limits_{x \to b^-} \dfrac{f(x)}{g(x)} = l$.

If $l \neq 0, \infty$, then $\int_a^b f$ & $\int_a^b g$ converge or diverge together at b.

Proof: Suppose $l > 0$. Choose $\varepsilon > 0$ small s.t. $l - \varepsilon > 0$.

$\because \lim\limits_{x \to b^-} \dfrac{f(x)}{g(x)} = l$, $\exists\, c \in [a,b)$ s.t.

$$\left| \frac{f(x)}{g(x)} - l \right| < \varepsilon \quad \forall\, x \in (c,b).$$

$$\Rightarrow \quad l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon \quad \forall\, x \in (c,b).$$

$$\Rightarrow \quad (l-\varepsilon)\, g(x) < f(x) < (l+\varepsilon)\, g(x).$$

$\because l - \varepsilon > 0$ & $g(x) \geq 0$. $\forall\, x$, it follows that

$$0 \leq (l-\varepsilon)\, g(x) \leq f(x) \quad \forall\, x \in (c,b).$$

$\therefore$ By Comparison test, if $\int_a^b f$, or equivalently, if $\int_c^b f$ converges at b, then $(l-\varepsilon)\int_c^b g$, or equivalently,

$(l-\varepsilon)\int_a^b g$ Converges at b. $\Rightarrow$ $\int_a^b g$ Converges at b.

Now suppose $\int_a^b g$, or equiv., $\int_c^b g$ converges at b.

Again by $f(x) < (l+\varepsilon)\, g(x) \quad \forall\, x \in (c,b)$ &

by the comparison test, it follows that $\int_a^b f$ converges at b.

$\therefore\ \int_a^b f$ converges at $b \Leftrightarrow \int_a^b g$ converges at b.

$|||^{ly}$; $\int_a^b f$ diverges at $b \Leftrightarrow \int_a^b g$ diverges at b.

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eg: (1) $\displaystyle\int_0^1 \frac{\sin x}{x^2}\, dx.$ $\longleftarrow$ An I.I. at $x=0$.

Set $f(x) = \dfrac{\sin x}{x^2}$ & $g(x) = \dfrac{1}{x}$.

$$\therefore\ \frac{f(x)}{g(x)} = \frac{\sin x}{x}.$$

$$\text{Now } \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 > 0.$$

But $\displaystyle\int_0^1 g = \int_0^1 \frac{1}{x}\, dx$ diverges. $\longleftarrow$ why?

$\therefore$ By Limit comparison test, $\displaystyle\int_0^1 \frac{\sin x}{x^2}\, dx$ diverges.

(2) $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}-1}{x}\, dx.$ $\longleftarrow$ I.I. at $x=0$.

Set $f(x) = \dfrac{e^{\sqrt{x}}-1}{x}$ & $g(x) = \sqrt{x}$.

$$\therefore\ \frac{f(x)}{g(x)} = \frac{\sqrt{x}\,(e^{\sqrt{x}}-1)}{x} = \frac{\sqrt{x}\,(\sqrt{x} + h(x)\times x)}{x}$$

$\longrightarrow 1$ as $x \longrightarrow 0^+$.

why?

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$\because \int_0^1 \sqrt{x}\, dx$ Converges, by Limit - comparison test,

$$\int_0^1 \frac{e^{\sqrt{x}} - 1}{x}\, dx \quad \text{converges.}$$

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(3) $$\int_1^2 \frac{x^2+x+1}{(x^2-1)^{1/3}}\, dx.$$

$$f(x) := \frac{x^2+x+1}{(x^2-1)^{1/3}} \longleftarrow \text{Rational fn.}$$

$$g(x) := \frac{1}{(1-x)^{1/3}}. \longleftarrow \because \text{We can deal with } \frac{1}{x+1} \text{ factor.}$$

$$\therefore \lim_{x \to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x^2+x+1)}{(x+1)^{1/3}} = \frac{3}{2^{1/3}} > 0.$$

$$\text{Now} \quad \int_1^2 g = \int_1^2 \frac{1}{(1-x)^{1/3}}\, dx$$

$$= \int_0^1 \frac{1}{x^{1/3}}\, dx \longleftarrow \left\{ \begin{array}{l} 1-x \mapsto x. \\ \Rightarrow 1 \to 0 \\ \& \ 2 \to 1 \end{array} \right.$$

$$\because p = \tfrac{1}{3} < 1, \quad \int_1^2 g \ \text{Converges.}$$

$$\Rightarrow \int_1^2 \frac{x^2+x+1}{(x^2-1)^{1/3}} \quad \underline{\text{Converges.}}$$

$\boxed{\mathbb{Z}}$

Def: An I.I. $\int_a^b f$ is said to be absolutely convergent if the I.I. $\int_a^b |f|$ is convergent.

Thm: Absolute convergent $\Rightarrow$ convergent. $\leftarrow$ Q: "$\Leftarrow$"?

Proof. Let $\int_a^b f$ be an I.I. at $x = a$.

Now $-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in (a,b]$.

$\Longrightarrow \quad 0 \leq f(x) + |f(x)| \leq 2|f(x)| \quad \forall x \in (a,b]$.

$\because$ the I.I $\int_a^b |f|$ is ~~absolutely~~ conv. at a, by comparison ~~test~~, it follows that

$$\int_a^b \left(f(x) + |f(x)| \right) dx \quad \text{converges.}$$

Finally, $\forall\ a < c < b$, we have:

$$\int_c^b f = \int_c^b \Big((f(x) + |f(x)|) - |f(x)| \Big) dx$$

$$= \int_c^b (f(x) + |f(x)|)\, dx - \int_c^b |f(x)|\, dx$$

~~(… both I.I. exist)~~

$\because \int_a^b (f(x) + |f(x)|)\, dx$ & $\int_a^b |f(x)|\, dx$ both exists, it follows that

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$\int_a^b f$ also exists. ▨

eg: (Recall from page 72).

$$\int_0^1 \frac{\sin x}{x^2} \cdot dx.$$

$\because$ $\left|\frac{\sin x}{x^2}\right| \leq |\sin x|$ $\forall\ x \in (0,1]$

& $\int_0^1 |\sin x|\, dx$ is convergent, it follows that

$\int_0^1 \frac{\sin x}{x^2}$ is A.C. & hence convergent. ▨

— x —

Def: (I.I. of type II):

Fix $a \in \mathbb{R}$ & suppose $f \in \mathcal{R}[a,r]$ $\forall\ r > a$.

If $\lim_{r \to +\infty} \int_a^r f$ exists, then we say that

$\int_a^{\infty} f$ Converges & we write:

$$\int_a^{\infty} f = \lim_{r \to +\infty} \int_a^r f.$$

If $\lim_{r \to +\infty} \int_a^r f$ diverges, then we say $\int_a^{\infty} f$ diverges.

Illy
$$\int_{-\infty}^{b} f = \lim_{r \to -\infty} \int_{r}^{b} f . \quad \longleftarrow \text{if Converges.}$$

~~Def:~~
$$\cancel{\int_{-\infty}^{\infty} f := \int_{-\infty}^{c} f + \int_{c}^{\infty} f}$$

~~whenever the~~ R.

Def: Let $f \in \mathcal{R}[a,b]$ $\forall$ $a < b$ in $\mathbb{R}$. If $\exists$ $c \in \mathbb{R}$ s.t. both $\int_{-\infty}^{c} f$ & $\int_{c}^{\infty} f$ Converge, then we say that $\int_{-\infty}^{\infty} f$ Converges & write

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f . \qquad \text{---} \quad (*)$$

HW: If $\int_{-\infty}^{\infty} f$ Converges, then $(*)$ is independent of the Choice of $c \in \mathbb{R}$.

eg: (1) $\int_{0}^{\infty} \sin x \, dx$. Let $f(x) = \sin x$. $x \in [0, \infty)$.

Note that $f \in \mathcal{R}[0, R]$ $\forall R > 0$.

$$\therefore \lim_{R \to \infty} \int_{0}^{R} f = \lim_{R \to \infty} \int_{0}^{R} \sin x \, dx$$

$$= \lim_{R \to \infty} \left[1 - \cos R \right] .$$

OR Does not converges.

But this limit DNE! $\Rightarrow$ $\int_{0}^{\infty} \sin x \, dx$ diverges.

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(2) $\int_{-\infty}^{0} e^{-x}\,dx$.

Again, $e^{-x} \in \mathcal{R}[-R, 0] \quad \forall\ R > 0$.

Now, for $R>0$,

$$\int_{-R}^{0} e^{-x}\,dx = \int_{0}^{R} e^{x}\,dx \qquad [x \longrightarrow -x]$$

$$= e^{R} - 1.$$

$$\therefore \lim_{R\to\infty} \int_{-R}^{0} e^{-x}\,dx = \lim_{R\to\infty} (e^{R}-1) = \infty.$$

$\therefore \int_{-\infty}^{0} e^{-x}\,dx$ diverges.

(3) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

We observe: $$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{0}^{R} \frac{dx}{1+x^2}$$

$$= \lim_{R\to\infty} \left[\tan^{-1}(R) - \tan^{-1}(0) \right].$$

why? $$= \pi/2 - 0 = \pi/2.$$

Also, $$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{-R}^{0} \frac{dx}{1+x^2}.$$

$$= \lim_{R\to\infty} \int_{0}^{R} \frac{dx}{1+x^2}. = \pi/2.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi.$$

$\boxed{///}$

This is an even fn.

Fact: (HW). Let $a > 0$. Then

$$\int_a^\infty \frac{1}{x^p}\,dx$$

Converges $\forall\ p > 1$ & diverges $\forall\ p \leq 1$.

Notation: For a fixed $a \in \mathbb{R}$, we say $f \in \mathcal{R}[a,\infty)$ if $f \in \mathcal{R}[a,R]$ $\forall\ R > a$.

CAUTION !! We are not saying that f is Riemann integrable.

Thm: (Comparison test - II).

Let $a \in \mathbb{R}$, $f, g \in \mathcal{R}[a,\infty)$ & let

$$0 \leq f(x) \leq g(x) \quad \forall\ x \in [a,\infty).$$

(i) If $\int_a^\infty g$ Converges, then $\int_a^\infty f$ Converges.

(ii) If $\int_a^\infty g$ diverges, then $\int_a^\infty f$ diverges.

Proof: $\because$ $0 \leq f(x) \leq g(x)$ $\forall\ x \in [a,\infty)$ & $f, g \in \mathcal{R}[a,\infty)$, it follows that

$$0 \leq \int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx \qquad \forall\ t > a.$$

Set $F(t) := \int_a^t f(x)\,dx$ & $G(t) := \int_a^t g(x)\,dx$. $\forall\ t > a$.

$\therefore$ $F, G \in C(a,\infty)$ & monotonically increasing.

The result now follows immediately.

(Cont. of F & G is not needed).

$\blacksquare$