Riemann Integration.

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CALCULUS

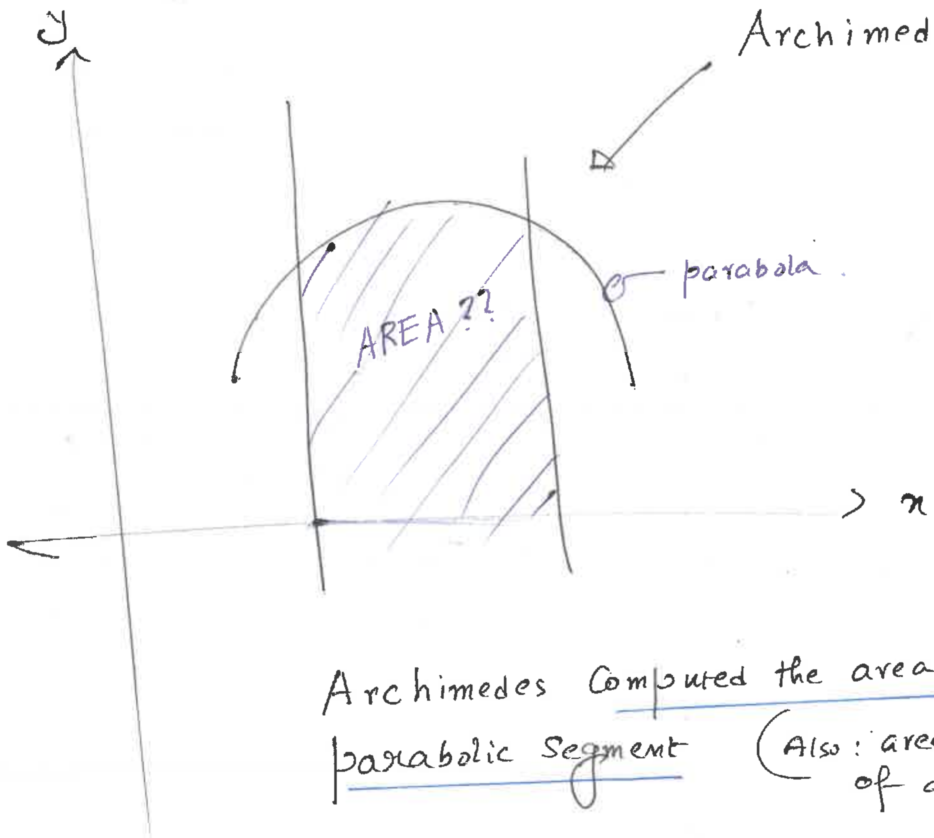
An old subject modified during 450 BC – till date.

"meaning": came from Latin:- Small stone / pebbel.

Started by ("perhaps") : Antiphone (430 BC. Greece).

Euclid (300 BC, Alexandria)

Archimedes (250 BC, Greece).



Archimedes computed the area of a parabolic segment (Also: area & circumference of a circle).

Then he asked:

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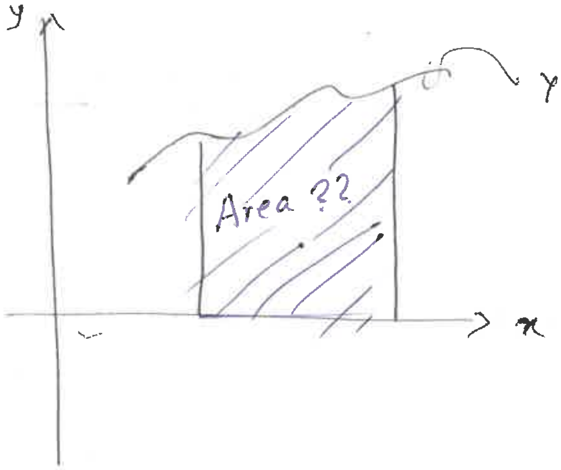
Area of :



$y = f(x)$
a "continuous" fn.

Ans: (Newton & ~~...~~ Leibniz : 1670). !!

He needs it for his law of motion.

This integration is/was fine : AND that's our "School integration."

It is still the "BEST" ! BUT, Conceptually.

∴ ① School/Newton/Leibniz integration: $\{ f : [a,b] \rightarrow \mathbb{R}$ Cont. fn $\}$.

② Riemann integration : An integration which make sense for bdd & not too "badly" discont. fn.

[He needed it for integrations of discont. fn (not too many discont.), to study Fourier Series.]

Present interest

③ Lebesgue integration : Deals with "highly" discont. & unbdd fns.

In master's : measure theory.

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Remember the following:

CALCULUS

Differential Calculus: $\frac{d}{dx}$

"Rough"

(diff. works for restricted fn's. Like, Continuity is a must for differentiability).

Cuts into small pieces (in microscopic level) & then finds out how it changes!

Integral Calculus: $\int$.

"Smooth".

(Int. works with lesser restricted fn's.)

joins the small pieces together to find out how much is there!!

CUT & GLUE

Fundamental theorem of Calculus. — Newton!!

"Rough & Smooth":

$\int$ makes fn's "Smoother".

$\frac{d}{dx}$ — " — "rougher"

eg: Indefinite integration of a cont. fn exists & differentiable. Whereas, a diff. fn's ~~derivativ~~ derivative need not be even cont!!

eg? — HW —.

Let's Start:

Always assume that $[a,b]$: a closed interval, $a<b$.

$\mathbb{N} = \{1, 2, \dots\}$.

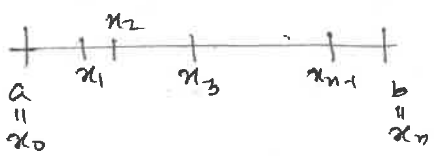
$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

$f : [a,b] \longrightarrow \mathbb{R}$ is ALWAYS bounded (stated otherwise).

Def: A partition P of $[a,b]$ is a set of real no's (called nodes) $\{x_0, x_1, \dots, x_n\}$, for some $n \in \mathbb{N}$ $\ni$.

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$



We ~~let~~ denote by I_j or P_j as $[x_{j-1}, x_j]$; the j-th subinterval $\forall j = 1, \dots, n$.

Def: If $I = (a,b)$ or $[a,b]$ or $[a,b)$ or $(a,b]$, then

$$|I| := b - a.$$

The length of I.

Fact: If $P : a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $I = [a,b]$, then

$$|I| = \sum_{j=1}^{n} |I_j| .$$

— HW —

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Remark: Let P & $\tilde{P}$ be two partitions of $[a,b]$. Then $P \cup \tilde{P}$ is also a partition of $[a,b]$.

[$P = \{x_0, x_1, \cdots, x_n\}$, $\tilde{P} = \{t_0, t_1, \cdots, t_m\}$, then

$$P \cup \tilde{P} = \{x_0, \cdots, x_n, t_0, \cdots, t_m\}$$

BUT ordered.]

eg: $I = [0,1]$. Then $\underset{x_0}{0} < \underset{x_1}{\frac{1}{8}} < \underset{x_2}{\frac{1}{4}} < \underset{x_3}{\frac{3}{4}} < \underset{x_4}{1}$ is a partition of I.

eg: $I = [0,1]$. Then $P = \{\frac{1}{n}\}_{n \geq 1} \cup \{0\}$ is NOT a partition of P. — Why?

Notation: 1) $\mathcal{P}[a,b] = \mathcal{P}$ = the set of all partitions of $[a,b]$.

2) Suppose $f : [a,b] \longrightarrow \mathbb{R}$ be a (BOUNDED) fn.

Let $P \in \mathcal{P}$ with

$$P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We set: $M_j := \sup_{x \in I_j} f(x)$ & $m_j := \inf_{x \in I_j} f(x)$,

$\forall j = 1, \cdots, n$.

Also, $M := \sup_{x \in [a,b]} f(x)$ & $m := \inf_{x \in [a,b]} f(x)$.

Observation: Let $S_1 \subseteq S_2 \subseteq \mathbb{R}$. Then
bdd.

$$\sup S_1 \leq \sup S_2 \quad \& \quad \inf S_1 \geq \inf S_2 .$$

— HW —.

Corollary: $m \leq m_j \leq M_j \leq M \qquad \forall\, j = 1, \dots, n.$

Notation: $\mathcal{B}[a,b]$ = Set of all bdd fn's $: [a,b] \longrightarrow \mathbb{R}$.

Def: Let $f \in \mathcal{B}[a,b]$ & let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. Then the upper Riemann sum of f w.r.t. P is defined by:

$$U(f;P) := \sum_{j=1}^{n} M_j \, |I_j| \qquad \left(= \sum_{j=1}^{n} \left(\sup_{x \in I_j} f(x) \right) \times (x_j - x_{j-1}) \right)$$

& the lower Riemann sum (w.r.t. P)

$$L(f;P) = \sum_{j=1}^{n} m_j \, |I_j| \qquad \left(= \sum_{j=1}^{n} \left(\inf_{x \in I_j} f(x) \right) \times (x_j - x_{j-1}) \right).$$

Note: Clearly, both $U(f;P)$ & $L(f;P)$ exist!!
[$\because f \in \mathcal{B}[a,b]$ & # nodes of $P < \infty$].

Thm: Given $f \in \mathcal{B}[a,b]$ ~~& $f \in \mathcal{B}[a,b]$~~, we have

$$m(b-a) \leq L(f;P) \leq U(f;P) \leq M(b-a).$$

$$\forall\, P \in \mathcal{P}[a,b].$$

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Proof: Fix $P: a = x_0 < \cdots < x_n = b$

Now ~~(b-a)~~ $m \leq m_j \leq M_j \leq M \quad \forall\, j = 1, \cdots, n$

$$\Rightarrow \quad m\,|I_j| \leq m_j\,|I_j| \leq M_j\,|I_j| \leq M\,|I_j| \quad \forall\, j = 1, \cdots, n.$$

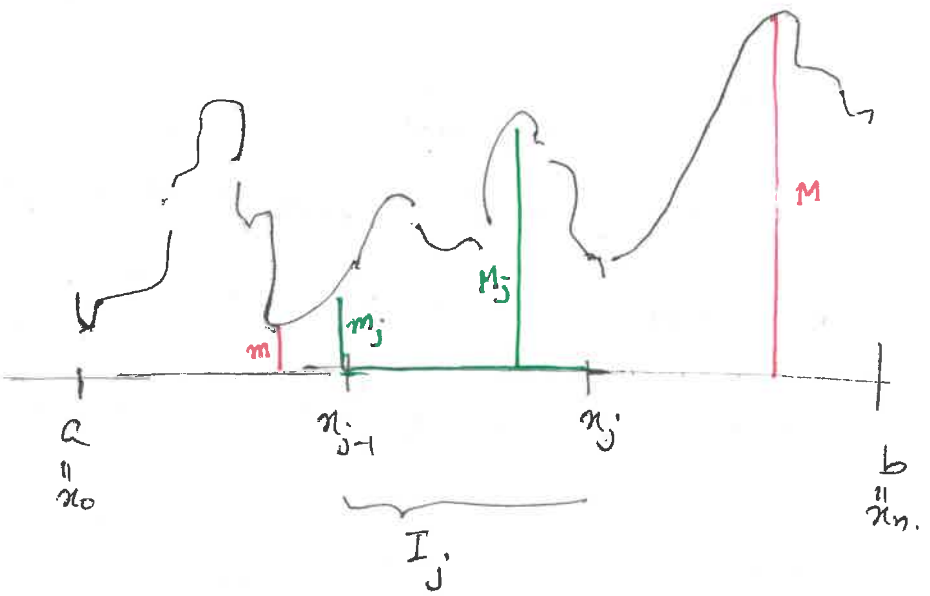
$[\because\ |I_j| > 0]$.

$$\Rightarrow \quad \underbrace{\sum_{j=1}^{m} m\,|I_j|}_{= m(b-a)} \leq \sum_{j=1}^{n} m_j\,|I_j| \leq \sum_{j=1}^{n} M_j\,|I_j| \leq \underbrace{\sum_{j=1}^{n} M\,|I_j|}_{= M(b-a)}.$$

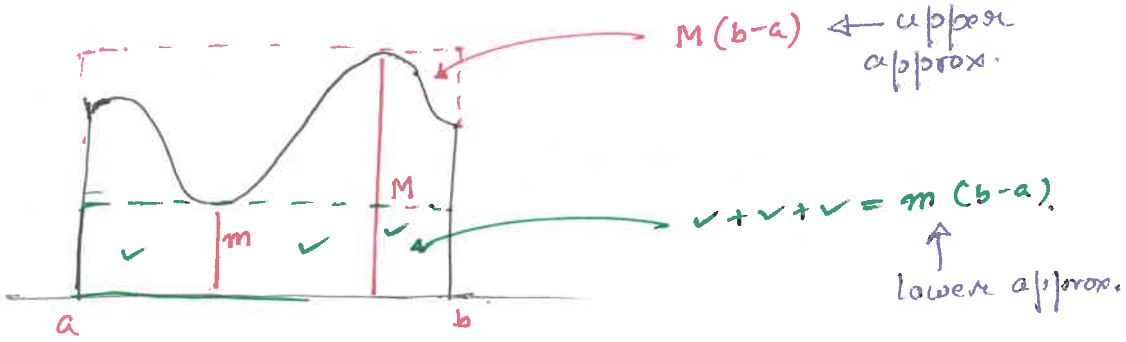
$$\Rightarrow \quad m(b-a) \leq L(f; P) \leq U(f; P) \leq M(b-a).$$

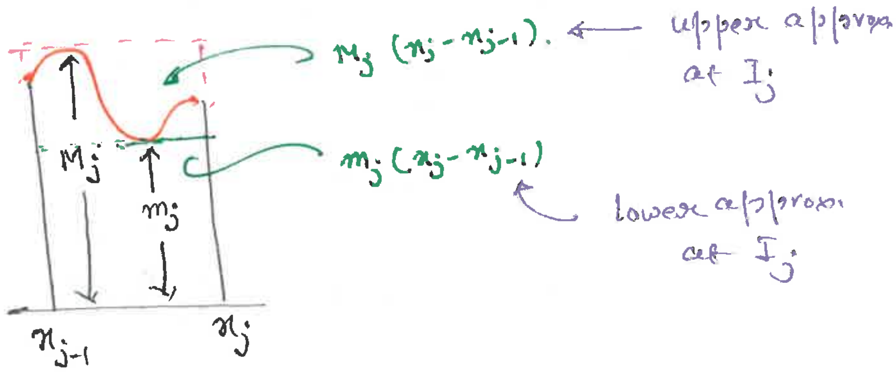
$\boxtimes$

The geometric implication (OR interpretation) of the above is pretty simple:



More simple example:





AND, Finally:

$$m(b-a) \leq L(f;P) \leq U(f;P) \leq M(b-a). \quad \forall P.$$

Independent of P

Independent of P.

$$\Downarrow$$

$\forall P \in \mathcal{P}[a,b]$, both $L(f;P)$ & $U(f;P)$

are bdd by $m(b-a)$ & $M(b-a)$.

i.e. $L(f;P), U(f;P) \in [m(b-a), M(b-a)]$.

The next idea would be: more nodes & increase $L(f,P)$.

— " — decrease $U(f,P)$.

& merge ↑ & ↓ at a point,

"If Possible".

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In view of "$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$", we are ready for the following ~~defini~~ definition: —— (*)

Def: Let $f \in \mathcal{B}([a,b])$. Define:

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \in \mathcal{P}[a,b]\}.$$

← Call it "The lower Riemann integration"

&

$$\overline{\int_a^b} f = \inf\{U(f,P) : P \in \mathcal{P}[a,b]\}.$$

← Call it "The upper Riemann integration".

Remark: (*) $\Rightarrow$ $\underbrace{\underline{\int} f}_{= \underline{\int_a^b} f}$ & $\underbrace{\overline{\int} f}_{= \overline{\int_a^b} f}$ both exist !!

Def: A fn $f \in \mathcal{B}[a,b]$ is said to be Riemann integrable, if

$$\underline{\int} f = \overline{\int} f \; ;$$

In this case, we call the common value as "the integration of f over $[a,b]$" & write:

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

← Good! But how to use? We need tools!!

Wish list: We really want "$L(f,P) \leq U(f,Q)$, $\forall P \,\&\, Q$ in $\mathcal{P}[a,b]$" to be true!!

⇝ & Indeed!!

Def: Let $P, \tilde{P} \in \mathcal{P}[a,b]$. We say that $\tilde{P}$ is a refinement of P [OR, $\tilde{P}$ is finer than P] if

$$x \in \tilde{P} \quad \forall \; x \in P.$$

$$\Updownarrow$$

nodes of $\tilde{P}$ $\supseteq$ nodes of P.

$\Uparrow$

i.e. $\tilde{P} = P \cup \hat{P}$, where $\hat{P}$ is a finite subset of $[a,b]$.

\# Notation: $\tilde{P} \supset P$ if $\tilde{P}$ is a refinement of P.

eg:

$$P = \{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\} \subset \tilde{P} = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\}.$$

Note: Let $P_1, P_2 \in \mathcal{P}[a,b]$. Set $\tilde{P} := P_1 \cup P_2$.

Then $\tilde{P} \supset P_1$ & $\tilde{P} \supset P_2$.

Proof: Trivial!!

Proposition: Let $f \in \mathcal{B}[a,b]$ & $P, \widetilde{P} \in \mathcal{P}[a,b]$. If $\widetilde{P} \supset P$, then

$$L(f,P) \leq L(f,\widetilde{P}) \leq U(f,\widetilde{P}) \leq U(f,P).$$

getting more ~~closer!(~~ closer!!

Proof: "$L(f,\widetilde{P}) \leq U(f,\widetilde{P})$" is known.

$\therefore$ Enough to prove "$L(f,P) \leq L(f,\widetilde{P})$" &

"$U(f,\widetilde{P}) \leq U(f,P)$".

We only prove the 1st one (as the 2nd one ~~will~~ be similar).

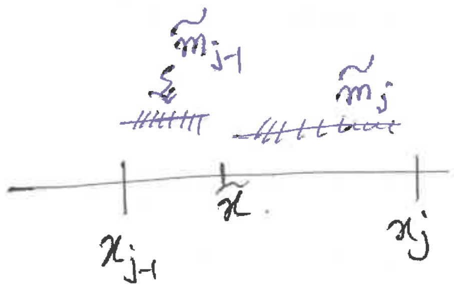
First, assume that $\widetilde{P} := P \cup \{\tilde{x}\}$,

~~Where~~ where $\tilde{x} \in [a,b] \setminus P$. [$\because$ $\tilde{x}$ a new node.]

Set $P: a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

Then $\exists$ $j \in \{1, \cdots, n\}$ s.t.

$$x_{j-1} < \tilde{x} < x_j. \quad \longleftarrow [\because x \in [a,b] \setminus \{a,b\}]$$

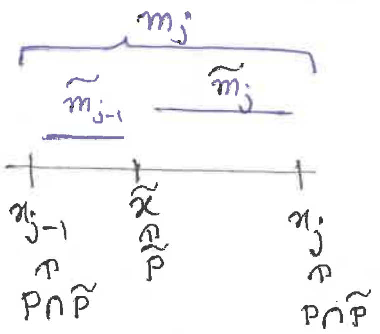


Set: $\tilde{m}_{j-1} := \inf\{f(x) : x \in [x_{j-1}, \tilde{x}]\}$

& $\tilde{m}_j := \inf\{f(x) : x \in [\tilde{x}, x_j]\}$.

$$\therefore\ L(f,\tilde{P}) - L(f,P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - x_{j-1}).$$

Now



$\Longrightarrow\ m_j \leq \tilde{m}_{j-1},\ \tilde{m}_j.$

$\because$ inf over ~~Sub~~ "smaller" subsets.

$$\Rightarrow L(f,\tilde{P}) - L(f,P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(\tilde{x} - x_{j-1}) - m_j(x_j - \tilde{x}).$$

$$\left[\because\ x_j - x_{j-1} = (x_j - \tilde{x}) + (\tilde{x} - x_{j-1})\right]$$

$$= \underbrace{(\tilde{m}_{j-1} - m_j)}_{\geq 0}(\tilde{x} - x_{j-1}) + \underbrace{(\tilde{m}_j - m_j)}_{\geq 0}(x_j - \tilde{x})$$

$$\geq 0$$

$\Rightarrow\ L(f,\tilde{P}) \geq L(f,P).$ The general case: by induction.

The upper sum case: Similar & HW. ☑

Cor: Let $f \in \mathcal{B}[a,b]$ & $P, Q \in \mathcal{P}[a,b]$. Then
$$L(f,P) \leq U(f,Q).$$

Proof: Let $\tilde{P} := P \cup Q$. $\Rightarrow \tilde{P} \supset P, Q$.

$$\therefore\ L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,Q).$$

{By applying the above prop. for $\tilde{P}, P$ & $\tilde{P}, Q$.}

In particular: $L(f,P) \leq U(f,Q)$. ☑

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Cor: If $f \in \mathcal{B}[a,b]$, then
$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof. We know: $L(f, P_1) \leq U(f, P_2) \quad \forall P_1, P_2 \in \mathcal{P}[a,b]$.

$\therefore$ For a fixed $P_2 \in \mathcal{P}[a,b]$,
$$\underline{\int_a^b} f = \sup_{P_1 \in \mathcal{P}[a,b]} L(f, P_1) \leq U(f, P_2).$$

$\therefore$ Taking inf on all over P_2 $\Rightarrow$ $\underline{\int} f \leq \inf_{P_2} U(f, P_2) = \overline{\int} f$.

$\boxtimes$

Notation: $\mathcal{R}[a,b] = \{ f \in \mathcal{B}[a,b] : f \text{ is Riemann integrable} \}$.

Fact: Suppose $f \in \mathcal{B}[a,b]$. Then
$$f \in \mathcal{R}[a,b] \iff \underline{\int_a^b} f \geq \overline{\int_a^b} f.$$

Q: $\mathcal{B}[a,b] = \mathcal{R}[a,b]$?

Ans: No!

eg: Consider the Dirichlet fn.: $f : [0,1] \to \mathbb{R}$ defined by:
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Clearly, $f \in \mathcal{B}[0,1]$.

Suppose $P : 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$.

Recall: $I_j := [x_{j-1}, x_j]$.

$\Rightarrow \quad I_j \cap \mathbb{Q} \neq \varphi \quad \& \quad I_j \cap \mathbb{Q}^c \neq \varphi.$

$\forall j = 1, \dots, n.$

$\Rightarrow \quad m_j = 0 \quad \& \quad M_j = 1 \qquad \forall j = 1, \dots, n.$

$\therefore \; L(f, P) = 0 \quad \& \quad U(f, P) = 1.$ [By the defn's. of L & U].

$\forall \; P \in \mathcal{P}[0,1]$.

$$\Rightarrow \quad \underline{\int_0^1} f = 0 \neq 1 = \overline{\int_0^1} f.$$

$\Rightarrow \quad f \notin \mathcal{R}[0,1]$. ▨

eg: ($\mathcal{R}[a,b] \neq \varphi$):

Fix $c \in \mathbb{R}$ & define $f(x) = c \quad \forall x \in [a,b]$.

Then, $\forall P \in \mathcal{P}[a,b]$, $\quad L(f,P) = c \times (b-a) = U(f,P)$.

Why? check.

$$\Rightarrow \quad \underline{\int_a^b} f = c \times (b-a) = \overline{\int_a^b} f$$

$\Rightarrow \quad f \in \mathcal{R}[a,b]$ & $\int_a^b f = c(b-a)$. ▨

eg: $\exists f$ s.t. $|f| \in \mathcal{R}[a,b]$ but $f \notin \mathcal{R}[a,b]$.

Consider $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$.

Clearly, $f \in \mathcal{B}[0,1]$. Here $|f| \equiv 1 \Rightarrow |f| \in \mathcal{R}[0,1]$.

But $f \notin \mathcal{R}[0,1]$, $\longleftarrow$ HW. ▨