Jaydeb Sarkar

Proposition: Let $f \in \mathcal{B}[a,b]$ & $P, \widetilde{P} \in \mathcal{P}[a,b]$. If $\widetilde{P} \supset P$, then

$$L(f,P) \leq L(f,\widetilde{P}) \leq U(f,\widetilde{P}) \leq U(f,P).$$

getting more ~~closer,(?)~~ closer!!

Proof: "$L(f,\widetilde{P}) \leq U(f,\widetilde{P})$" is known.

$\therefore$ Enough to prove "$L(f,P) \leq L(f,\widetilde{P})$" & "$U(f,\widetilde{P}) \leq U(f,P)$".

We only prove the 1^{st} one (as the 2^{nd} one ~~will~~ be similar).

First, assume that $\widetilde{P} := P \cup \{\tilde{x}\}$,

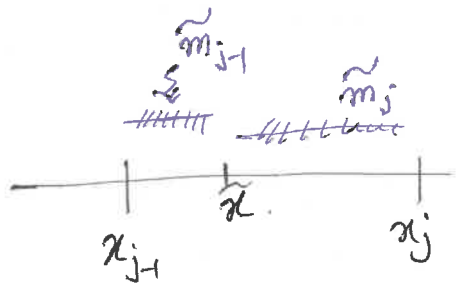
~~Where~~ where $\tilde{x} \in [a,b] \setminus P$. [$\therefore$ $\tilde{x}$ a new node.]

Set $P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

Then $\exists\ j \in \{1, \cdots, n\}$ s.t.

$$x_{j-1} < \tilde{x} < x_j.$$

$\leftarrow$ [$\because$ $x \in [a,b] \setminus \{a,b\}$]

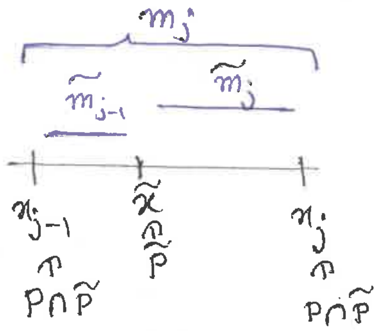


Set: $\tilde{m}_{j-1} := \inf\{f(x) : x \in [x_{j-1}, \tilde{x}]\}$

& $\tilde{m}_j := \inf\{f(x) : x \in [\tilde{x}, x_j]\}$.

$$\therefore \; L(f,\tilde{P}) - L(f,P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(x_j - x_{j-1}).$$

Now



$\Longrightarrow \; m_j \leq \tilde{m}_{j-1}, \tilde{m}_j$.

$\because$ inf over ~~sub~~ "smaller" subsets.

$$\Rightarrow L(f,\tilde{P}) - L(f,P) = \tilde{m}_{j-1}(\tilde{x} - x_{j-1}) + \tilde{m}_j(x_j - \tilde{x}) - m_j(\tilde{x} - x_{j-1}) - m_j(x_j - \tilde{x}).$$

$$\left[\because x_j - x_{j-1} = (x_j - \tilde{x}) + (\tilde{x} - x_{j-1})\right]$$

$$= \underbrace{(\tilde{m}_{j-1} - m_j)}_{\geq 0}(\tilde{x} - x_{j-1}) + \underbrace{(\tilde{m}_j - m_j)}_{\geq 0}(x_j - \tilde{x})$$

$$\geq 0$$

$\Rightarrow \; L(f,\tilde{P}) \geq L(f,P)$.

The general case: by induction.

The upper sum case: Similar & HW.

$\boxtimes$

Cor: Let $f \in \mathcal{B}[a,b]$ & $P, Q \in \mathcal{P}[a,b]$. Then

$$L(f,P) \leq U(f,Q).$$

Proof: Let $\tilde{P} := P \cup Q$. $\Rightarrow \tilde{P} \supset P, Q$.

$$\therefore \; L(f,X) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,X)$$

(By applying the above prop. for $\tilde{P}, P$ & $\tilde{P}, Q$.) where $X = P$ & Q.

In particular: $L(f,P) \leq U(f,Q)$. $\boxtimes$

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Cor: If $f \in \mathcal{B}[a,b]$, then
$$\underline{\int_a^b} f \leq \overline{\int_a^b} f .$$

Proof: We know: $L(f, P_1) \leq U(f, P_2) \quad \forall P_1, P_2 \in \mathcal{P}[a,b]$.

$\therefore$ For a fixed $P_2 \in \mathcal{P}[a,b]$,
$$\underline{\int_a^b} f = \sup_{P_1 \in \mathcal{P}[a,b]} L(f, P_1) \leq U(f, P_2).$$

$\therefore$ Taking inf on all over P_2 $\Rightarrow$ $\underline{\int} f \leq \inf\limits_{P_2} U(f, P_2) = \overline{\int} f$.

$\boxtimes$

Notation: $\mathcal{R}[a,b] = \{ f \in \mathcal{B}[a,b] : f \text{ is Riemann integrable.} \}$.

Fact: Suppose $f \in \mathcal{B}[a,b]$. Then
$$f \in \mathcal{R}[a,b] \Leftrightarrow \underline{\int_a^b} f \geq \overline{\int_a^b} f .$$

Q: $\mathcal{B}[a,b] = \mathcal{R}[a,b]$?

Ans: No!

eg: Consider the Dirichlet fn.: $f : [0,1] \to \mathbb{R}$ defined by:
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1] . \end{cases}$$

Clearly, $f \in \mathcal{B}[0,1]$.

Suppose $P : 0 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$.

Recall: $I_j := [x_{j-1}, x_j]$.

$$\Rightarrow \quad I_j \cap \mathbb{Q} \neq \varphi \quad \& \quad I_j \cap \mathbb{Q}^c \neq \varphi. \quad \forall j = 1, \dots, n.$$

$$\Rightarrow \quad m_j = 0 \quad \& \quad M_j = 1 \qquad \forall j = 1, \dots, n.$$

$$\therefore \quad L(f, P) = 0 \quad \& \quad U(f, P) = 1. \quad [\text{By the defn's. of } L \,\&\, U].$$

$$\forall \; P \in \mathcal{P}[0,1].$$

$$\Rightarrow \quad \underline{\int_0^1} f = 0 \neq 1 = \overline{\int_0^1} f.$$

$$\Rightarrow \quad f \notin \mathcal{R}[0,1].$$

$\boxtimes$

eg: ($\mathcal{R}[a,b] \neq \varphi$):

Fix $c \in \mathbb{R}$ & define $f(x) = c \quad \forall x \in [a,b]$.

Then, $\forall P \in \mathcal{P}[a,b]$, $\quad L(f, P) = c \times (b-a) = U(f, P)$.

Why? check.

$$\Rightarrow \quad \underline{\int_a^b} f = c \times (b-a) = \overline{\int_a^b} f$$

$$\Rightarrow \quad f \in \mathcal{R}[a,b] \quad \& \quad \int_a^b f = c(b-a).$$

$\boxtimes$

eg: $\exists f$ s.t. $|f| \in \mathcal{R}[a,b]$ but $f \notin \mathcal{R}[a,b]$.

Consider $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$.

Clearly, $f \in \mathcal{B}[0,1]$. Here $|f| \equiv 1 \Rightarrow |f| \in \mathcal{R}[0,1]$.
But $f \notin \mathcal{R}[0,1]$, $\leftarrow$ HW.

$\boxtimes$