Jaydeb Sarkar

Note

In view of the above example (i.e. $f \notin \mathcal{R}[a,b]$ but $|f| \in \mathcal{R}[a,b]$), we have the following question:

$$f \in \mathcal{R}[a,b] \overset{?}{\Longrightarrow} |f| \in \mathcal{R}[a,b] \quad ?$$

— Think about it —.

Tool

Thm: Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff$ for $\varepsilon > 0$ $\exists$ $P \in \mathcal{P}[a,b]$ s.t

$$U(f,P) - L(f,P) < \varepsilon .$$

Proof: "$\Leftarrow$" Let $\varepsilon > 0$.

$\therefore$ $\exists$ $P \in \mathcal{P}[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Now $$L(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f,P).$$

(true in general)

$$\Rightarrow \overline{\int_a^b} f \leq U(f,P) < \varepsilon + L(f,P).$$

(By assumption)

But $$L(f,P) \leq \underline{\int_a^b} f$$

$$\therefore \overline{\int_a^b} f < \varepsilon + \underline{\int_a^b} f$$

$$\Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon .$$

We also know, in general, that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

$$\therefore 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon . \quad \forall \varepsilon > 0.$$

$$\Rightarrow \underline{\int} f = \overline{\int} f \Rightarrow f \in \mathcal{R}[a,b].$$

"$\Rightarrow$" Suppose $f \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$.

$\therefore \exists\, P_1 \in \mathcal{P}[a,b] \ni$ $\quad L(f, P_1) > \underline{\int_a^b} f - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2}$. —— (A)

& $\exists\, P_2 \in \mathcal{P}[a,b] \ni$ $\quad U(f, P_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}$. —— (B)

Set $P := P_1 \cup P_2$. $\Rightarrow P \supset P_1$ & P_2.

Claim: $U(f,P) - L(f,P) < \varepsilon$.

Now, ~~Now~~, $U(f, P) \underset{(\because\, P \supset P_2)}{\leq} U(f, P_2) \underset{(B)}{<} \int_a^b f + \frac{\varepsilon}{2}$.

$$\underset{(A)}{<} L(f, P_1) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\underset{(\because\, P \supset P_1)}{\leq} L(f, P) + \varepsilon.$$

$\Rightarrow U(f,P) - L(f,P) < \varepsilon$. $\boxed{}$

Note: We always have the following:

$$0 \leq U(f,P) - L(f,P) \quad !!$$

Def: Let $P \in \mathcal{P}[a,b]$. Then the mesh of P (or norm of P) is defined as:

$$\|P\| := \max\{x_j - x_{j-1} : 1 \leq j \leq n\},$$

where $P: a = x_0 < x_1 < \cdots < x_n = b$.

Jaydeb Sarkar

A kind of "continuity" property of Riemann integ..

Thm: (Darboux thm) Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

$$\forall\ P \in \mathcal{P}[a,b] \text{ with } \|P\| < \delta.$$

[For $f \in \mathcal{B}[a,b]$ fixed, define $\eta : \mathcal{P}[a,b] \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\eta(P) = U(f,P) - L(f,P) \quad \forall\ P \in \mathcal{P}[a,b].$$

So, $f \in \mathcal{R}[a,b] \iff$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\eta(P) < \varepsilon \quad \forall\ \|P\| < \delta \quad !! \;]$$

Proof: "$\Leftarrow$" : Follows from ~~prev~~ the last observation.

"$\Rightarrow$" Let $f \in \mathcal{R}[a,b]$ & let $\varepsilon > 0$.

~~Goal: $\exists\ P \in \mathcal{P}[a,b] \ni U(f,P) - L(f,P) < \varepsilon$.~~

~~[Again, by the prev. observation, we will conclude that~~

$\therefore$ $\exists$ $\tilde{P} \in \mathcal{P}[a,b]$ s.t. $U(f,\tilde{P}) - L(f,\tilde{P}) < \frac{\varepsilon}{2}$. $\quad (*)$

(By the prev. obs.)

~~Set~~ Assume that $\boxed{\#\text{ nodes of } \tilde{P} = p.}$

$\because$ $f \in \mathcal{B}[a,b]$, $\exists$ $M > 0$ s.t.

$$|f(x)| \leq M \quad \forall\ x \in [a,b]$$

Our friendly bound.

Set $\quad \delta := \dfrac{\varepsilon}{8pM}$. $\longleftarrow$ Suggested by a back calculation.

Let $P \in \mathcal{P}[a,b]$ & suppose $\underbrace{\|P\|}_{\text{max length of subintervals}} < \delta$.

Set $\hat{P} := P \cup \tilde{P}$ $\longleftarrow$ $\therefore\ \hat{P} \supset P, \tilde{P}$.

$\Rightarrow$ $\hat{P}$ has atmost p nodes that are not in P.

Now, ~~if~~ let $\hat{P} = P \cup \{\tilde{x}\}$ & $\tilde{x} \notin P$. [i.e. $p=1$ case].

As earlier: Set $P: a = x_0 < x_1 < \cdots < x_n = b$.

(p-12)

& assume $x_{j-1} < \tilde{x} < x_j$.

Then:
$$L(f,\hat{P}) - L(f,P) = (\tilde{m}_{j-1} - m_j)(\tilde{x} - x_{j-1}) + (\tilde{m}_j - m_j)(x_j - \tilde{x})$$

[See page - 12]

$$\leq 2M\|P\|.$$

Illy, if $\hat{P} = P \cup \{\tilde{x}_1, \ldots, \tilde{x}_p\}$, then (by induction),

$$L(f,\hat{P}) - L(f,P) \leq 2Mp\|P\|$$
$$< 2Mp \times \delta = \frac{\varepsilon}{4}.$$

$$\left.\begin{array}{l} \therefore\ L(f,\hat{P}) - L(f,P) < \frac{\varepsilon}{4} \\ \text{Illy}\quad U(f,P) - U(f,\hat{P}) < \frac{\varepsilon}{4} \end{array}\right\}$$ —— (†) (**)

$\therefore$ (†) $\Rightarrow$ $U(f,P) - L(f,P) < \frac{\varepsilon}{2} + \left(U(f,\hat{P}) - L(f,\hat{P})\right)$ —— (**)

But (*) $\Rightarrow$ $U(f,\tilde{P}) - L(f,\tilde{P}) < \frac{\varepsilon}{2}$. ~~& (**)~~ $\because\ \hat{P} \supset \tilde{P}$, we know

$L(f,\tilde{P}) \leq L(f,\hat{P})$ & $U(f,\tilde{P}) \geq U(f,\hat{P})$.

$\Longrightarrow$ $U(f,\hat{P}) - L(f,\hat{P}) \leq U(f,\tilde{P}) - L(f,\tilde{P}) < \varepsilon/2$. ~~& (**) ⇒ U(f,P̃) − L(f,P̂~~

$\therefore$ (**) $\Rightarrow$ $U(f,P) - L(f,P) < \varepsilon$. ☒

Notation: $C[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} \text{ continuous} \}$.

\# $\{\text{polynomials}\} \subseteq C[a,b]$. AND $\{\text{rationals}\} \subseteq C[a,b]$.

Also: $\{ e^x, \sin x, \cos x, \dots \} \subseteq C[a,b]$.

Who are they ??

$\therefore$ This is a large class !!

Any relation $\{e^x, \sin x, \cos x, \dots\} \overset{??}{\longleftrightarrow} \mathbb{R}[x]$??

A pending question.

Thm: $C[a,b] \subseteq R[a,b]$.

Proof: Let $f \in C[a,b]$.

$\Rightarrow$ $f : [a,b] \longrightarrow \mathbb{R}$ is uniformly continuous.

Let $\varepsilon > 0$. By uniform cont. $\exists$ $\delta > 0$ s.t.

$$| f(x) - f(y) | < \underbrace{\frac{\varepsilon}{b-a}}_{\text{The new "}\varepsilon\text{"}} \quad \forall\, x, y \in [a,b] \text{ s.t. } |x-y| < \delta.$$

Let $P \in \mathcal{P}[a,b]$ & assume $\|P\| < \delta$.

Set $P : a = x_0 < x_1 < \cdots < x_n = b$.

Now $f|_{[x_{j-1}, x_j]} : [x_{j-1}, x_j] \longrightarrow \mathbb{R}$ is also unif. cont. $\forall j = 1, \dots, n$.

$\Rightarrow$ $f|_{[x_{j-1}, x_j]}$ assumes it max (which is M_j) & min (which is m_j) in $[x_{j-1}, x_j]$ $\forall j = 1, \dots, n$.

$\because$ $\|P\| < \delta$, we know

$$x_j - x_{j-1} < \delta \quad \forall\ j = 1, \dots, n.$$

The max length of subintervals

In particular: $|x-y| < \delta \quad \forall\ x, y \in [x_{j-1}, x_j]$.

$$\therefore \quad M_j - m_j < \frac{\varepsilon}{b-a} \quad \forall\ j = 1, \dots, n.$$

$$\therefore\ U(f,P) - L(f,P) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) - \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$

$$= \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1})$$

$$< \frac{\varepsilon}{b-a} \times \sum_{j=1}^{n} (x_j - x_{j-1})$$

$$= \frac{\varepsilon}{b-a} \times b-a$$

$$= \varepsilon.$$

$$\Rightarrow U(f,P) - L(f,P) < \varepsilon.$$

$$\Rightarrow f \in \mathcal{R}[a,b]. \quad \boxtimes$$

Q: Still, how to compute $\int_a^b f$ for $f \in C[a,b]$??

— WAIT —