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On "how to Compute $\int_a^b f$?"

1st attempt: (Sequential approach)

Thm: Let $f \in \mathcal{B}[a,b]$. Then $f \in \mathcal{R}[a,b] \iff \exists$ a seqn. $\{P_n\} \subseteq \mathcal{P}[a,b]$ s.t.

$$\lim_{n \to \infty} \left[U(f, P_n) - L(f, P_n) \right] = 0.$$

Moreover, in this case:

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n \to \infty} L(f, P_n).$$

Proof: [Note: $\because$ $U(f,P) - L(f,P) \geq 0 \quad \forall P \in \mathcal{P}[a,b]$,
$\{U(f,P_n) - L(f,P_n)\} \subseteq \mathbb{R}_{\geq 0}$. $\circledast$].

"$\Rightarrow$" Let $f \in \mathcal{R}[a,b]$. We know: "for $\varepsilon > 0$, $\exists$ $P_\varepsilon \in \mathcal{P}[a,b]$ $\ni$ $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$." ← Call it "Cauchy Criterion"

$\therefore$ For each $n \in \mathbb{N}$, $\exists$ $P_n \in \mathcal{P}[a,b]$ s.t.

$$U(f, P_n) - L(f, P_n) < \frac{1}{n}.$$

By $\circledast$ above: $U(f, P_n) - L(f, P_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

"$\Leftarrow$" If $\lim_{n\to\infty} \left[U(f,P_n) - L(f,P_n)\right] = 0$, then for $\varepsilon > 0$

$\exists$ $N \in \mathbb{N}$ $\ni$ $U(f, P_N) - L(f, P_N) < \varepsilon$.

$\Longrightarrow f \in \mathcal{R}[a,b]$. By Cauchy Criterion.

Finally, if $f \in \mathcal{R}[a,b]$, then ~~for~~ $\{P_n\} \subseteq \mathcal{P}[a,b]$ as above:

We have:

$$0 \leqslant U(f, P_n) - \overline{\int_a^b} f \qquad \longleftarrow \left(\because \overline{\int} f = \inf U(f,P)\right)$$

$$= U(f, P_n) - \underline{\int_a^b} f \qquad \longleftarrow \because f \in \mathcal{R}[a,b]$$

$$\left(\text{As } \underline{\int} f = \sup_P L(f,P)\right) \rightarrow \leqslant U(f, P_n) - L(f, P_n) \longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} U(f, P_n) = \overline{\int_a^b} f = \int_a^b f.$$

Finally,

$$L(f, P_n) = U(f, P_n) - \big(U(f, P_n) - L(f, P_n)\big)$$

$$\downarrow$$

$$\int_a^b f - 0 \qquad \text{as } n \to \infty$$

Standard argument.

$$\Rightarrow \lim_{n\to\infty} L(f, P_n) = \int_a^b f.$$

$\boxtimes$

DANGER: $a_n - b_n \longrightarrow 0 \not\Longrightarrow$ Even $\lim a_n$ & / or $\lim b_n$ exist !!

Remark: Evidently, if $\exists \{P_n\} \subseteq \mathcal{P}[a,b]$ s.t.
$L(f, P_n) \longrightarrow c$ & $U(f, P_n) \longrightarrow c$, then

$$f \in \mathcal{R}[a,b] \quad \& \quad \int_a^b f = c.$$

A nice way to prove existence of $\mathcal{R}$-integrability (& evaluating too) of bdd fn's. ["Reminding us Newton"].

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eg: $f(x) = x^2$. $x \in [0,1]$.

[BTW: $f \in C[0,1]$
$\Rightarrow f \in \mathcal{R}[0,1]$.
But we prove it directly.]

Fix $n \in \mathbb{N}$.

Consider "the classical" partition:

Why call it classical?

$$P_n : \quad 0 = x_0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = 1.$$

$$\therefore \quad I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right] \quad \forall\ j = 1, \ldots, n.$$

$\because f$ is $\uparrow$, it follows that:

$$m_j = \left(\frac{j-1}{n}\right)^2 \quad \& \quad M_j = \left(\frac{j}{n}\right)^2.$$

$\forall\ j = 1, \ldots, n$.

$$\therefore\ U(f, P_n) = \sum_{j=1}^{n} M_j \times \frac{1}{n} = \sum_{j=1}^{n} \frac{j^2}{n^2} \times \frac{1}{n}.$$

$$= \frac{1}{n^3} \sum_{j=1}^{n} j^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) = \frac{1}{n^3} \times \frac{1}{6} \times n \times (n+1)(2n+1)$$

$$= \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Also, $$L(f, P_n) = \sum_{j=1}^{n} m_j \frac{1}{n} = \frac{1}{n^3} \sum_{j=1}^{n} (j-1)^2.$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right)$$

$$= \frac{1}{n^3} \times \frac{1}{6} \times (n-1) \times n \times \left(2(n-1) + 1\right)$$

$$= \frac{1}{6} \times \left(1 - \frac{1}{n}\right) \times \left(2 - \frac{1}{n}\right).$$

$$\therefore\ U(f, P_n) \longrightarrow \frac{1}{3} \quad \& \quad L(f, P_n) \longrightarrow \frac{1}{3}.$$

$\Rightarrow \quad U(f, P_n) - L(f, P_n) \longrightarrow 0.$

$\Rightarrow \quad \underline{f \in \mathcal{R}[0,1]}$ & $\underline{\int_0^1 f = \lim_{n\to\infty} U(f, P_n) = \frac{1}{3}}$. ▨

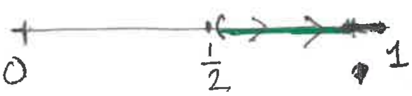
Q: $\mathcal{R}[a,b] \setminus C[a,b] \neq \varphi$?

i.e. $\exists\ f \in \mathcal{B}[a,b]$ s.t. $f \in \mathcal{R}[a,b]$ but $f \notin C[a,b]$??

Ans: Yes.

eg: (1)

$$f(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & \frac{1}{2} < x \le 1. \end{cases}$$



Claim: $f \in \mathcal{R}[0,1]$. Clearly, $\underline{f \notin C[0,1]}$.

Why? "ε-δ" argument?

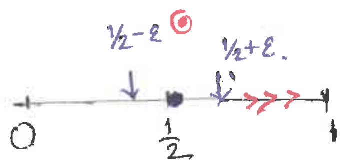
Proof

Let $\varepsilon > 0$. (Small).

Consider the partition: $P_\varepsilon: \ 0 = x_0 < x_1 = \frac{1}{2} - \varepsilon < x_2 = \frac{1}{2} + \varepsilon < x_3 = 1$

i.e: $0 < \frac{1}{2} - \varepsilon < \frac{1}{2} + \varepsilon < 1.$

A 2-node partition.



$\therefore \quad \left.\begin{array}{l} I_1 = [0, \frac{1}{2} - \varepsilon] \\ I_2 = [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \\ I_3 = [\frac{1}{2} + \varepsilon, 1] \end{array}\right\}$

$\therefore \quad \left.\begin{array}{l} m_1 = 1, \ m_2 = 0, \ m_3 = 0 \\ M_1 = 1, \ M_2 = 1, \ M_3 = 0 \end{array}\right\}$.

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$$\therefore \quad L(f, P_\varepsilon) = \sum_{j=1}^{3} m_j \, |I_j|$$

$$= 1 \times \left(\tfrac{1}{2} - \varepsilon\right) + 0 + 0.$$

$$= \tfrac{1}{2} - \varepsilon.$$

& $$U(f, P_\varepsilon) = \sum_{j=1}^{3} M_j \, |I_j|$$

$$= 1 \times \left(\tfrac{1}{2} - \varepsilon\right) + 1 \times 2\varepsilon + 0.$$

$$= \tfrac{1}{2} + \varepsilon.$$

$$\therefore \quad U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon.$$

i.e. $U(f, P_\varepsilon) - L(f, P_\varepsilon) < 3\varepsilon.$

why? $\Rightarrow f \in \mathcal{R}[0,1].$

Q: $\int_0^1 f = ?$

$\therefore$ For $\varepsilon = \frac{1}{2n}$, we have,

$U(f, P_n) - L(f, P_n) = \frac{1}{n} \to 0.$

$\Rightarrow \lim \left[U(f, P_n) - L(f, P_n)\right] = 0.$

$\Rightarrow \int_0^1 f = \frac{1}{2}.$

Remark: In fact, if $f \in \mathcal{B}[0,1]$ with finitely many discontinuity, then $f \in \mathcal{R}[0,1]$.

— WAIT —.

— × —.

Let's make it more refined !!

The theory

Def: Let $P: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$. ~~Define~~ A fn.

$$T_P : \{I_j\}_{j=1}^{n} \longrightarrow [a,b] \quad \text{is called a tag of } P \text{ if}$$

$$T_P(I_j) \in I_j \qquad \forall\, j=1,\dots,n.$$

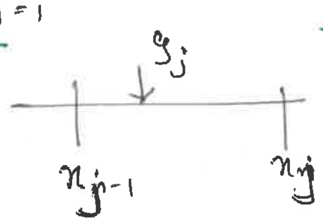
Simply

Given a partition $P: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$, a tag or tag set of P is a collection of points $\{\xi_j\}_{j=1}^{n}$ s.t.

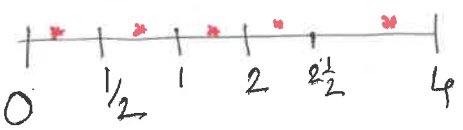
$$\xi_j \in I_j \qquad \forall\, j=1,\dots,n.$$

Even Simpler.

A tag set $T_P=\{\xi_j\}_{j=1}^{n}$ of P/ for P/ corresponding to P. where $\xi_j\in I_j$ $\forall\, j=1,\dots,n$.



eg:



$$P: 0<\tfrac{1}{2}<1<2<2\tfrac{1}{2}<4$$

$$T_P=\left\{\tfrac{1}{6},\ \tfrac{5}{8},\ \tfrac{4}{3},\ 2\tfrac{1}{5},\ 3\tfrac{1}{3}\right\}$$

a tag set.

Note: If P has n-nodes [nodes excludes end points $\Rightarrow (n+1)$ subintervals], then $\#T_P=n+1$ $\forall$ tag set T_P.

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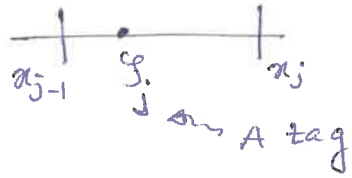
Def: Let $f \in \mathcal{B}[a,b]$, $P \in \mathcal{P}[a,b]$. & suppose T_P a tag set of P.

The Riemann sum of f w.r.t. (P, T_P) is defined by:

$$S(f, P) := \sum_{j=1}^{n} f(\xi_j)\, |I_j| \, , \quad \text{———} \quad \circledast$$

where: $T_P = \{\xi_j\}_{j=1}^{n}$.



Note: ① $S(f, P)$ depends on $T_P = \{\xi_j\}_{j=1}^{n}$.

② LHS of $\circledast$ doesn't involve T_P but it is there !!

③ $\exists$ infinitely many tag sets for a given partition P.

④ In fact a question; What is the meaning of Riemann sum ?

Fact: (Answering ④)

Suppose $P : a = x_0 < \cdots < x_n = b$ be a partition &

$T_P = \{\xi_j\}_{j=1}^{n}$ be a tag of P.

$$\therefore \quad m_j \leq f(\xi_j) \leq M_j \qquad \forall\, j = 1, \dots, n$$

$$\Longrightarrow \quad m_j\, |I_j| \leq f(\xi_j)\, |I_j| \leq M_j\, |I_j| \qquad \forall\, j = 1, \dots, n.$$

$$\Longrightarrow \quad L(f, P) \leq S(f, P) \leq U(f, P).$$

$\therefore$ it is a better approximation.

HOWEVER, $S(f, P)$ depends on T_P but $L(f, P)$ & $U(f, P)$ does not !!