Jaydeb Sarkar.

Abel's lemma: Let $\{a_j\}_{j=1}^{n}$ be a set of decreasing +ve numbers (i.e. $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$) & $\{w_j\}_{j=1}^{n}$ be a set of real nos.

Suppose $\alpha \leq \sum_{j=1}^{m} w_j \leq \beta \quad \forall\, m = 1, \cdots, n$,

for some $\alpha, \beta \in \mathbb{R}$.

Then $a_1 \alpha \leq \sum_{j=1}^{n} a_j w_j \leq a_1 \beta$.

~~Proof~~ simply [i.e. If $\alpha \leq \sum_{j=1}^{m} w_j \leq \beta \quad \forall\, m = 1, \cdots, n$, then $\forall$ decreasing $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have:

$$a_1 \alpha \leq \sum_{j=1}^{n} a_j w_j \leq a_1 \beta.\ ]$$

Proof: Set $S_m := \sum_{j=1}^{m} w_j \quad \forall\, m = 1, \cdots, n$. m-th partial sum.

We know $\alpha \leq S_m \leq \beta \quad \forall\, m = 1, \cdots, n$.

Now
$$\sum_{j=1}^{n} a_j w_j = a_1 S_1 + a_2 (S_2 - S_1) + \cdots + a_n (S_n - S_{n-1})$$
$$= (a_1 - a_2) S_1 + (a_2 - a_3) S_2 + \cdots + \cdots + (a_{n-1} - a_n) S_{n-1} + a_n S_n. \quad (*)$$

$\because a_j - a_{j+1} \geq 0 \quad \forall\, j = 1, \cdots, n-1$ & $S_m \leq \beta \quad \forall\, m = 1, \cdots, n$,

by $(*)$, we have
$$\sum_{j=1}^{n} a_j w_j \leq \beta \left[(a_1 - a_2) + \cdots + (a_{n-1} - a_n) + a_n \right] = \beta a_1.$$

& since $\alpha \leq S_m \quad \forall\, m$, by $(*)$ we have:
$$\sum_{j=1}^{m} a_j w_j \geq \alpha \left[(a_1 - a_2) + \cdots + (a_{n-1} - a_n) + a_n \right] = \alpha a_1.$$

$$\therefore \alpha a_1 \leq \sum_{j=1}^{m} a_j w_j \leq \beta a_1. \qquad \boxtimes$$

Now we are ready for the 2^{nd} MVT. The finer version is due to Weierstrass. First we prove the "initial variant":

Thm: (2^{nd} MVT : Bonnet's form) :

Let $f, \varphi \in \mathcal{R}[a,b]$, and suppose $\varphi \geq 0$ & monotonically decreasing on $[a,b]$. Then $\exists\ \xi \in [a,b]$ s.t.

$$\int_a^b \varphi f = \varphi(a) \int_a^{\xi} f .$$

← f is a kind of "weight fn"?

Proof: Let $P \in \mathcal{P}[a,b]$. & assume $P: a = x_0 < x_1 < \cdots < x_n = b$.

Pick $\xi_j \in I_j \quad \forall j = 2, \ldots, n$ & $\xi_1 := a$.

$\therefore \{\xi_j\}_{j=1}^n$ is a tag set of P.

Recall: $I_j = [x_{j-1}, x_j]$, $j = 1, \ldots, n$. & $m_j = \inf_{I_j} f$, $M_j = \sup_{I_j} f$

We know:

$$m_j (x_j - x_{j-1}) \leq \int_{x_{j-1}}^{x_j} f \leq M_j (x_j - x_{j-1})$$

$$\& \quad m_j (x_j - x_{j-1}) \leq f(\xi_j)(x_j - x_{j-1}) \leq M_j (x_j - x_{j-1})$$

$\forall j = 1, \ldots, n$.

By taking t-th partial sums

$$\Longrightarrow \sum_{j=1}^{t} m_j |I_j| \leq \int_a^{x_t} f \leq \sum_{j=1}^{t} M_j |I_j|$$

$$\& \quad \sum_{j=1}^{t} m_j |I_j| \leq \sum_{j=1}^{t} f(\xi_j) |I_j| \leq \sum_{j=1}^{t} M_j |I_j|$$

$\forall\ t = 1, \ldots, n$.

Combining above pair of inequalities

$$\Longrightarrow \left| \int_a^{x_t} f - \sum_{j=1}^{t} f(\xi_j) |I_j| \right| \leq \sum_{j=1}^{t} (M_j - m_j) |I_j|$$

$$\leq \sum_{j=1}^{n} (M_j - m_j) |I_j| \qquad \forall\ t = 1, \ldots, n.$$

Jaydeb Sarkar

$$\Rightarrow \left| \int_a^{x_t} f - \sum_{j=1}^{t} f(\xi_j)|I_j| \right| \leq \sum_{j=1}^{n} (M_j - m_j)|I_j|.$$

$$\Longleftrightarrow \int_a^{x_t} f - \sum_{j=1}^{n}(M_j - m_j)|I_j| \leq \underbrace{\cancel{\int_a^{x_t} f}}_{\sum_{j=1}^{t} f(\xi_j)|I_j|} \leq \int_a^{x_t} f + \sum_{j=1}^{n}(M_j - m_j)|I_j|$$

$\forall\, t = 1, \cdots, n.$ — (†)

Now we observe that $x \longmapsto \int_a^x f(t)\,dt$ is a cont. fn. on $[a,b]$ $(\because f \in \mathcal{R}[a,b])$.

$\Rightarrow$ In particular:

$$\delta_1 := \min_{x \in [a,b]} \int_a^x f \leq \int_a^y f \leq \delta_2 := \sup_{x \in [a,b]} \int_a^x f.$$

$\forall\, y \in [a,b].$

$$\therefore (\dagger) \Rightarrow \delta_1 - \underbrace{\sum_{j=1}^{n} (M_j - m_j)|I_j|}_{:= \underset{P}{\mathrm{osc}}\, f \,=\, U(f,P) - L(f,P)} \leq \sum_{j=1}^{t} f(\xi_j)|I_j| \leq \delta_2 + \sum_{j=1}^{n}(M_j - m_j)|I_j|.$$

$\forall\, t = 1, \cdots, n.$

Set $a_j := \varphi(\xi_j)$ $\quad j = 1, \ldots, n.$

$\because \varphi \geq 0$ & $\varphi \downarrow$

By assumption: $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0.$

Therefore, we are in the setting of Abel's lemma, with:

$$\left\{ \begin{array}{l} \alpha := \delta_1 - \underset{P}{\mathrm{osc}}\, f \leq \underbrace{\sum_{j=1}^{t} f(\xi_j)|I_j|}_{:= \omega_t} \leq \beta := \delta_2 + \underset{P}{\mathrm{osc}}\, f. \quad \forall\, t = 1, \cdots, n. \\ \& \quad a_1 \geq a_2 \geq \cdots \geq a_n. \end{array} \right.$$

$\therefore$ By Abel's lemma:

$$a_1 \alpha \leq \sum_{j=1}^{n} a_j \omega_j \leq a_1 \beta.$$

i.e. $$\varphi(a)\, \alpha \leq \sum_{j=1}^{n} \varphi(\xi_j) f(\xi_j)\, |I_j| \leq \varphi(a)\, \beta.$$

$$[\because\ a_1 = \varphi(\xi_1) = \varphi(a)]$$

Note that: $$\operatorname*{osc}_P f = \sum_{j=1}^{m} (M_j - m_j)\, |I_j|$$

$$= U(f,P) - L(f,P).$$

The Riemann Sum

$$\therefore\ \varphi(a) \times \left[\delta_1 - \big(U(f,P) - L(f,P)\big) \right] \leq R(\varphi f, P) \leq \varphi(a) \times \left[\delta_2 + \big(U(f,P) - L(f,P)\big) \right].$$

$$\forall\ P \in \wp[a,b].$$

$\because$ δ_1, δ_2 are independent of P, as $\|P\| \to 0$,

$$\varphi(a)\, \delta_1 \leq \int_a^b \varphi f \leq \varphi(a)\, \delta_2.$$

$$\left[\because\ \|P\| \to 0 \Rightarrow U(f,P) - L(f,P) \to 0 \ \ \& \ \ R(\varphi f, P) \to \int_a^b \varphi f.\right]$$

But $\delta_1 = \min_{x\in[a,b]} \int_a^x f$ & $\delta_2 = \max_{x\in[a,b]} \int_a^x f$. & $x \mapsto \int_a^x f$ is cont.

$$\Rightarrow \int_a^b \varphi f = \varphi(a) \int_a^{\xi} f. \text{ for some } \xi \in [a,b].$$

$\blacksquare$

Jaydeb Sarkar

Thm: (2nd MVT : Weierstrass' form).

Let $f, \varphi \in \mathcal{R}[a,b]$ & φ is monotonic on $[a,b]$. Then $\exists\, \xi \in [a,b]$ s.t.

$$\int_a^b \varphi f = \varphi(a)\int_a^{\xi} f + \varphi(b)\int_{\xi}^b f.$$

Proof: WLOG: assume φ is $\uparrow$ [otherwise, consider $-\varphi$].

Set $\tilde{\varphi}(x) := \varphi(x) - \varphi(b)$ $\forall\, x \in [a,b]$.

$\therefore$ $\tilde{\varphi} \in \mathcal{R}[a,b]$, $\tilde{\varphi} \geq 0$ & $\tilde{\varphi}$ monotonically decreasing on $[a,b]$. By 2nd MVT, Bonnet's form, $\exists\, \xi \in [a,b]$ s.t.

$$\int_a^b \tilde{\varphi} f = \tilde{\varphi}(a)\int_a^{\xi} f.$$

$$\Rightarrow \int_a^b \varphi f - \varphi(b)\int_a^b f = \left(\varphi(a) - \varphi(b)\right)\int_a^{\xi} f.$$

$$\Rightarrow \int_a^b \varphi f = \varphi(a)\int_a^{\xi} f + \varphi(b)\left[\int_a^b f - \int_a^{\xi} f\right]$$

$$= \varphi(a)\int_a^{\xi} f + \varphi(b)\int_{\xi}^b f.$$

$\blacksquare$

— x —

Back to Type II improper integration:

We want to prove two tests:

Thm (Abel's test): Let $\varphi \in B[a,\infty)$ be a monotonic fn. & let $\int_a^\infty f$ converges. Then $\int_a^\infty \varphi f$ also converges.

Proof: We know $f \in \mathcal{R}[a,R]$ $\forall R > a$. Let $a < R_1 < R_2$. By 2nd MVT (Weierstrass version), $\exists$ $\xi \in [R_1, R_2]$ s.t.

$$\int_{R_1}^{R_2} \varphi f = \varphi(R_1)\int_{R_1}^{\xi} f + \varphi(R_2)\int_{\xi}^{R_2} f \quad \text{——} \ (*)$$

Let $M := \sup_{x \in [a,\infty)} |\varphi(x)|$, & let $\varepsilon > 0$.

$\because \int_a^\infty f$ converges, $\exists$ ~~$R_0 \in \mathbb{R}$ s.t.~~ $R_0 \in \mathbb{R}$ s.t.

$$\left| \int_{B_1}^{B_2} f \right| < \varepsilon/2M \quad \forall B_1, B_2 \geq R_0 \quad \text{——} \ (**)$$

Cauchy criterion/test

Assume $R_1, R_2 \geq R_0$. Then $|\varphi(R_1)|, |\varphi(R_2)| \leq M$.

& hence, $(*)$ & $(**)$ $\Longrightarrow$

$$\left| \int_{R_1}^{R_2} \varphi f \right| \leq |\varphi(R_1)| \left| \int_{R_1}^{\xi} f \right| + |\varphi(R_2)| \left| \int_{\xi}^{R_2} f \right|$$

$$\leq M \times \frac{\varepsilon}{2M} + M \times \frac{\varepsilon}{2M} = \varepsilon.$$

$$\Rightarrow \left| \int_{R_1}^{R_2} \varphi f \right| < \varepsilon \quad \forall R_1, R_2 \geq R_0.$$

$\Rightarrow \int_a^\infty \varphi f$ converges (by Cauchy criterion/test).

$\square$

Jaydeb Sarkar.

Thm: (Dirichlet test):

Let $\varphi \in \mathcal{B}[a,\infty)$ be a monotonic fn. & $\lim_{x \to \infty} \varphi(x) = 0$.

Suppose $f \in \mathcal{R}[a,\infty)$ & $x \longmapsto \int_a^x f$ is a bdd fn. on $[a,\infty)$. Then $\int_a^\infty \varphi f$ Converges.

Proof. Let $M := \sup_{x \in [a,\infty)} \left| \int_a^x f \right|$.

Let $\varepsilon > 0$. As $\lim_{x \to \infty} \varphi(x) = 0$, $\exists\, m_0 \in \mathbb{R}$ s.t.

$$|\varphi(x)| < \varepsilon/4M \quad \forall\, x > m_0.$$

Suppose $R_1, R_2 > m_0$.

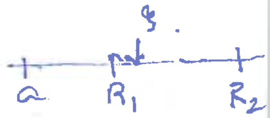
By 2nd MVT (of Weierstrass form), $\exists\, \xi$ between R_1 & R_2 s.t.

$$\left| \int_{R_1}^{R_2} \varphi f \right| = \left| \varphi(R_1) \int_{R_1}^{\xi} f + \varphi(R_2) \int_{\xi}^{R_2} f \right|$$

$$\leq |\varphi(R_1)| \left| \int_{R_1}^{\xi} f \right| + |\varphi(R_2)| \left| \int_{\xi}^{R_2} f \right|$$

$$< \frac{\varepsilon}{4M} \times \left(\left| \int_{R_1}^{\xi} f \right| + \left| \int_{\xi}^{R_2} f \right| \right).$$

$\because |\varphi(x)| < \frac{\varepsilon}{4M}$ $\forall\, x > m_0$. & $R_1, R_2 > m_0$.



Now $\left| \int_{R_1}^{\xi} f \right| = \left| \int_a^{\xi} f - \int_a^{R_1} f \right|$

$$\leq \left| \int_a^{\xi} f \right| + \left| \int_a^{R_1} f \right|$$

$$\leq 2M \qquad \left[\because M = \sup_{x \in [a,\infty)} \left| \int_a^x f \right| \right]$$

Ily $\left| \int_{\xi}^{R_2} f \right| \leq 2M.$

$\therefore \left| \int_{R_1}^{R_2} \varphi f \right| < \frac{\varepsilon}{4M}\left(2M + 2M \right) = \varepsilon.$

i.e. $\left| \int_{R_1}^{R_2} \varphi f \right| < \varepsilon \quad \forall\ R_1, R_2 > m_0.$

$\Rightarrow \int_a^{\infty} \varphi f$ Converges. [by Cauchy test].

$\boxtimes$

eg: $\int_1^{\infty} \frac{1}{x} \sin x \cdot \log x \cdot dx.$

Set $f(x) = \sin x$, $\varphi(x) = \frac{\log x}{x}$.

Now $\int_1^x \sin t\, dt = \cos 1 - \cos x \Rightarrow \left| \int_1^x f \right| \leq 2 \quad \forall x \in [1,\infty).$
$\forall x \in [1,\infty)$

$\Rightarrow \sup_{x \in [1,\infty)} \left| \int_1^x f \right| \leq 2.$

Also, $\varphi(x) \downarrow$ & $\varphi(x) \to 0$ as $x \to \infty$ [Why?].

$\therefore$ By Dirichlet test, $\int_1^{\infty} \varphi f$ Converges. $\boxtimes$