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eg: ** $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$.

Set $f(x) = \sin x$. We already know:

$$\left|\int_1^x f\right| \leq 2 \quad \forall\, x \in [1, \infty)$$

Also, if $\varphi(x) := \dfrac{1}{x^p}$, then $\varphi \downarrow$ & $\varphi(x) \longrightarrow 0$ as $x \to \infty$ whenever $p > 0$.

$\therefore$ By Dirichlet test

$$\int_1^\infty \frac{\sin x}{x^p}\,dx \text{ Converges } \forall\, p > 0. \quad (*)$$

Remark:

# $\because$ $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ Converges, in particular,

$$\int_0^\infty \frac{\sin x}{x}\,dx \text{ Converges.} \qquad \left[\because \int_0^\infty = \int_0^1 + \int_1^\infty\right].$$

Q: What about AC? —WAIT—

# $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ is A.C. whenever $p > 1$.

Indeed: $\left|\dfrac{\sin x}{x^p}\right| \leq \dfrac{1}{x^p} \quad \forall\, x \geq 1$

& $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ Conv. whenever $p > 1$.

$\therefore$ By Comparison test: $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ is A.C. $\forall\, p > 1$.

# In fact, We know ~~Since~~ $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ Converges $\iff p > 1$,

~~by limit Comparison Test~~,

Note that $|\sin x| \geq \sin^2 x$.

So, if $0 < p \leq 1$, then:

$$\left|\frac{\sin x}{x^p}\right| \geq \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p} \quad (x \geq 1). \qquad \oplus$$

As in ⊛ above, by Dirichlet test, $\displaystyle\int_1^\infty \frac{\cos 2x}{2x^p}\,dx$ Converges $\forall\, p > 0$.

But $\displaystyle\int_1^\infty \frac{1}{2x^p}\,dx$ diverges $\forall\, p \leq 1$.

$$\Longrightarrow \int_1^\infty \frac{1-\cos 2x}{2x^p}\,dx \text{ diverges } \forall\, p \leq 1.$$

$\therefore$ By Comparison test:

$$\int_1^\infty \frac{\sin x}{x^p}\,dx \text{ diverges } \forall\, p \leq 1.$$

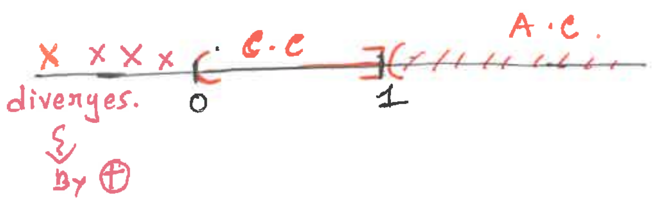
$$\therefore \int_1^\infty \frac{\sin x}{x^p}\,dx \text{ is A.C. } \iff p > 1.$$

Then, ⊛ $\Rightarrow$ $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ is Conditionally Converges (i.e. Converges but not A.C.)

for all $0 < p \leq 1$. ▨

— x —

So, for $\displaystyle\int_0^1 \frac{\sin x}{x^p}\,dx$,



We know: $\int_0^\infty \frac{\sin x}{x}\,dx$ Converges, BUT:

Known as the Dirichlet integral

**Thm:** $\int_0^\infty \frac{\sin x}{x}\,dx$ is Conditionally Converges.

**Proof:** All we need to prove is that $\int_0^\infty \frac{\sin x}{x}\,dx$ is NOT A.C.

(See Page-95)

Fix $n \in \mathbb{N}$. Set $f(x) = \frac{\sin x}{x}$

Now $$\int_0^{n\pi} |f| = \sum_{m=1}^{n} \int_{(m-1)\pi}^{m\pi} |f|$$

$\forall\, m = 1, \dots, n$, we have

$$\int_{(m-1)\pi}^{m\pi} |f| = \int_{(m-1)\pi}^{m\pi} \frac{|\sin x|}{x}\,dx. \qquad \because x \geq 0$$

$$= \int_0^{\pi} \frac{|\sin x|}{(m-1)\pi + x}\,dx. \qquad \text{using } \left(x \longmapsto (m-1)\pi + x.\right)$$

$$= \int_0^{\pi} \frac{\sin x}{(m-1)\pi + x}\,dx. \qquad \because \sin x \geq 0 \;\; \forall\, x \in [0,\pi].$$

Now $(m-1)\pi + x \leq m\pi \quad \forall\, x \in [0,\pi]$.

$$\Rightarrow \frac{1}{(m-1)\pi + x} \geq \frac{1}{m\pi} \quad \forall\, x \in [0,\pi].$$

$$\Rightarrow \frac{\sin x}{(m-1)\pi + x} \geq \frac{\sin x}{m\pi} \quad \forall\, x \in [0,\pi].$$

$$\therefore \int_{(m-1)\pi}^{m\pi} |f| \geq \frac{1}{m\pi} \int_0^{\pi} \sin x\,dx = \frac{1}{m\pi}\left(\cos 0 - \cos\pi\right)$$

$$= \frac{2}{m\pi} \quad \forall\, m = 1, \dots, n.$$

i.e. $$\int_{(m-1)\pi}^{m\pi} |f| \geq \frac{2}{m\pi} \quad \forall\ m = 1, \cdots, n.$$

$$\therefore \int_0^{n\pi} |f| = \sum_{m=1}^{n} \int_{(m-1)\pi}^{m\pi} |f|$$

$$\geq \frac{2}{\pi} \times \sum_{m=1}^{n} \frac{1}{m}. \qquad \forall\ n \in \mathbb{N}.$$

$\because \sum_{m=1}^{\infty} \frac{1}{m}$ is divergent, it follows that

$$\lim_{n\to\infty} \int_0^{n\pi} \left| \frac{\sin x}{x} \right| dx = \infty.$$

$$\Rightarrow \lim_{R\to\infty} \int_0^{R} \left| \frac{\sin x}{x} \right| dx = \infty.$$

Indeed, for $R \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t.

$$n\pi \leq R < (n+1)\pi.$$

$$\Rightarrow \int_0^{R} |f| \geq \int_0^{(n+1)\pi} |f|.$$

$\therefore$ By Comparison test, $\int_0^{R} |f| \longrightarrow \infty$.

$\boxed{///}$

HW: Use Similar method / tricks to prove that

$$\int_2^{\infty} \frac{\cos x}{\log x}\, dx \text{ is Conditionally Convergent.}$$

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eg: $\int_0^1 f$ is conditionally convergent, where

$$f(x) = \begin{cases} 0 & \text{if } x = 0. \\ (-1)^{n+1}(n+1) & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \end{cases}, \quad n = 1, 2, \ldots\ldots$$

Infinite step fn

Clearly $\int_0^1 |f|$ is an I.I. of type-I. lly $\int_0^1 f$.

Why? HW.

Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ $\ni$. $\varepsilon \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$

$1 > \varepsilon > 0$

i.e. $\frac{1}{n+1} < \varepsilon \leq \frac{1}{n}$.

Now
$$\int_\varepsilon^1 |f| = \int_\varepsilon^{1/n} + \int_{1/n}^{1/n-1} + \cdots + \int_{1/3}^{1/2} + \int_{1/2}^{1}$$

$$= \int_\varepsilon^{1/n} (n+1)\,dx + \int_{1/n}^{1/n-1} n\,dx + \cdots + \int_{1/3}^{1/2} 3\,dx + \int_{1/2}^{1} 2\,dx.$$

$$= (n+1) \times \left(\frac{1}{n} - \varepsilon\right) + n\left(\frac{1}{n-1} - \frac{1}{n}\right) + \cdots + 3 \times \left(\frac{1}{2} - \frac{1}{3}\right) + 2 \times \left(1 - \frac{1}{2}\right)$$

$$= (n+1)\left(\frac{1}{n} - \varepsilon\right) + \left\{\frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + 1\right\}.$$

$$= \sum_{m=1}^{n-1} \frac{1}{m} + \underbrace{(n+1)\left(\frac{1}{n} - \varepsilon\right)}_{>0}.$$

$\frac{1}{n+1}$ ( $\varepsilon$ ] $\frac{1}{n}$.

$$> \sum_{m=1}^{n-1} \frac{1}{m}.$$

$\because \sum_{m=1}^{\infty} \frac{1}{m}$ diverges, & $\varepsilon \to 0^- \Rightarrow n \to \infty$,

it follows that $\lim_{\varepsilon \to 0^-} \int_\varepsilon^1 |f|$ diverges.

On the other hand: for $\varepsilon > 0$, a similar calculation yields:

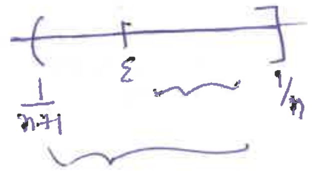
$$\int_\varepsilon^1 f = \int_\varepsilon^{1/n} (-1)^{n+1}(n+1)\,dx + \int_{1/n}^{1/n-1} (-1)^n n\,dx + \cdots + \int_{1/3}^{1/2} (-3)\,dx + \int_{1/2}^1 2\,dx.$$

$$= (-1)^{n+1}(n+1)\left(\frac{1}{n} - \varepsilon\right) + (-1)^n \frac{1}{n-1} + \cdots + \frac{1}{3} - \frac{1}{2} + 1.$$

$$\Rightarrow \int_\varepsilon^1 f - \sum_{m=1}^{n-1} (-1)^{m+1} \frac{1}{m} = (-1)^{n+1}(n+1)\left(\frac{1}{n} - \varepsilon\right).$$

$$\Rightarrow \left| \int_\varepsilon^1 f - \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{m} \right| = \left| (n+1)\left(\frac{1}{n} - \varepsilon\right) \right|$$

$$= (n+1) \times \left(\frac{1}{n} - \varepsilon\right)$$




$$< (n+1) \times \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n}.$$

$$\Rightarrow \left| \int_\varepsilon^1 f - \sum_{m=1}^{n-1} \frac{(-1)^{m+1}}{m} \right| < \frac{1}{n}.$$

$\because$ the alternating series $\sum_{m=1}^{\infty} (-1)^{m+1}/m$ converges,

& since $\varepsilon \to 0^- \Rightarrow n \to \infty$, it follows that

$$\lim_{\varepsilon \to 0^-} \int_\varepsilon^1 f = \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

$\therefore \int_0^1 f$ is conditionally convergent. $\boxtimes$

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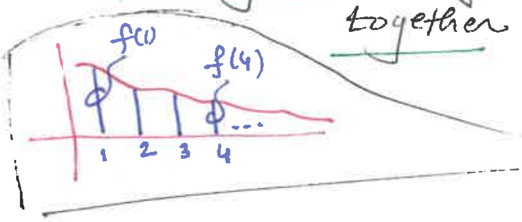
Similar idea applies to the following:

**Thm:** (Cauchy - MacLaurin test) Let $f \downarrow$, $f(x) > 0$ on $[1, \infty)$. Then $\int_1^\infty f$ & $\sum_{n=1}^\infty f(n)$ converge or diverge together.




**Proof:** Fix $n \in \mathbb{N}$. By assumption:

$$f(n) \geq f(x) \geq f(n+1) \quad \forall\ x \in [n, n+1].$$

$$\Rightarrow f(n) \geq \int_n^{n+1} f \geq f(n+1) \quad \forall\ n \in \mathbb{N}. \qquad \text{—— } \circledast.$$

Set $S_n := \sum_{m=1}^n f(m)$. $\forall\ n \in \mathbb{N}$ ← $n$-th partial sum.

$$\therefore \circledast \Rightarrow \quad S_{n-1} \geq \int_1^n f \geq \underbrace{\sum_{m=1}^{n-1} f(m+1)}_{= \sum_{m=2}^{n} f(m)}, \ \forall\ n \geq 2. \qquad \text{—— } (**)$$

$$\Rightarrow S_{n-1} \geq \int_1^n f \geq \sum_{m=2}^n f(m).$$

Observe that: "
~~Now~~ $\lim_{R \to \infty} \int_1^R f = \alpha \in \mathbb{R}$, $\Longleftrightarrow$ $\lim_{n \to \infty} \int_1^n f = \alpha$ " —— $(***)$

$\therefore$ If $\lim_{n \to \infty} \int_1^n f = \alpha$ (i.e. $\int_1^\infty f$ converges), then

$$(**) \Rightarrow 0 \leq \sum_{m=2}^n f(m) \leq \int_1^n f \leq \alpha \qquad \left[\because \int_1^n f \uparrow\right]$$

$$\Rightarrow \{S_n\}_{n=1}^\infty \text{ is bdd} \Rightarrow \sum_{n=1}^\infty f(n) \text{ converges.}$$

If $\int_1^\infty f$ diverges, ~~then~~ then using (**); i.e,

$$\int_1^n f \leq S_{n-1}$$

we have that $\{S_n\}_{n=1}^\infty$ unbounded ($\&$ $\uparrow$).

$\Rightarrow \sum_{n=1}^\infty f(n)$ diverges.

$\therefore \int_1^\infty f$ Converges (/ diverges) $\Rightarrow \sum_{n=1}^\infty f(n)$ Converges (/diverges).

Illy: (***) & (**) implies:

$$\sum_{n=1}^\infty f(n) \text{ Converges/div.} \Rightarrow \int_1^\infty f \text{ Conv./div.}$$

$\boxed{//}$

eg: p-series: $\sum_{n=1}^\infty \frac{1}{n^p}$, $p > 0$.

Consider $\int_1^\infty \frac{1}{x^p}\,dx$.

We know the above I.I. Conv. $\forall\ p > 1$ & div. $\forall\ p \leq 1$.

$\therefore$ The p-series conv. $\forall\ p > 1$ & div. $\forall\ 0 < p \leq 1$.

A known fact !!