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Gamma fn.:

Recall the notion of factorial: $n! = n(n-1)\cdots 3.2.1. \quad \forall\, n \in \mathbb{N}.$

Q: What about $x! = ?$ $\quad (x \in \mathbb{R}_{>0})$!!

The hint is (as we will also see soon): Also, a HW.

$$\int_0^\infty x^n e^{-x}\, dx = n! \qquad \forall\, n \geq 1.$$

We need to generalize this concept!!

Def:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt. \qquad \forall\, x > 0.$$

Euler's gamma fn. on $(0,\infty)$.

Thm: $\Gamma(x)$ Converges $\forall\, x > 0$.

Proof:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x \in (0,\infty)$

Two issues: $\underset{\text{Type I}}{0}$ & $\underset{\text{Type II}}{\infty}$.

$$= \underbrace{\int_0^1 t^{x-1} e^{-t}\, dt}_{:= \gamma_1(x)} + \underbrace{\int_1^\infty t^{x-1} e^{-t}\, dt}_{:= \gamma_2(x)}$$

For $\gamma_1(x) = \int_0^1 t^{x-1} e^{-t}\, dt$, we observe: For any $x > 0$,

$$0 < t^{x-1} e^{-t} < t^{x-1} \qquad \forall\, t \in (0,1].$$

$[\because e^{-t} < 1]$

$\because \int_0^1 t^{x-1}\,dt = \frac{1}{x}$, by Comparison test, it follows that $\gamma_1(x)$ Converges.

For $\gamma_2(x) = \int_1^\infty t^{x-1} e^{-t}\,dt$, $x > 0$,

We note $\lim_{t\to\infty}\left(t^{x+1} e^{-t}\right) = 0$ ← HW. (use L' Hospital rule a couple of times)

$\left[\lim_{x\to\infty} \frac{x^p}{e^x} = 0 \quad \forall\, p \in \mathbb{R}\right]$.

$\therefore \exists\, M > 1$ s.t. $t^{x+1} e^{-t} \leq 1 \quad \forall\, t \geq M$.

$$\Rightarrow \quad t^{x-1} e^{-t} \leq \frac{1}{t^2} \quad \forall\, t \geq M$$

But $\int_M^\infty \frac{1}{t^2}\,dt$ Converges ($p = 2$ case).

$\therefore$ By Comparison test : $\gamma_2(x) = \int_1^\infty t^{x-1} e^{-t}\,dt$ Converges $\forall\, x > 0$.

$$\int_1^\infty = \int_1^M + \int_M^\infty$$

$\Rightarrow \Gamma(x)$ Converges $\forall\, x > 0$. $\boxed{//}$

Thm: $\Gamma(x+1) = x\,\Gamma(x) \quad \forall\, x > 0$. ← Expected property.

~~Proof: $\forall\, a < b$, using by parts:~~

$$\cancel{\int_a^b t^{x-1} e^{-t}\,dt = -t^{x-1} e^{-t}\Big|_a^b + (x-1)\int_a^b t^{x-2} e^{-t}\,dt}$$

$$\cancel{= \underbrace{\left(a^{x-1} e^{-a} - b^{x-1} e^{-b}\right)}_{\downarrow \atop 0 \text{ as } a\to 0 \;\&\; b\to\infty} + (x-1)\int_a^b t^{x-2} e^{-t}\,dt.}$$

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Proof: Note that $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$.

Now $\forall\ r_1, r_2 > 0$, we have, by parts, that

$$\int_{r_1}^{r_2} t^x e^{-t}\,dt = -t^x e^{-t}\Big]_{t=r_1}^{r_2} + x\int_{r_1}^{r_2} t^{x-1} e^{-t}\,dt$$

$$= \underbrace{\left(\underbrace{r_1^x e^{-r_1}}_{0 \text{ as } r_1 \to 0} - \underbrace{r_2^x e^{-r_2}}_{0 \text{ as } r_2 \to \infty}\right)}_{0 \text{ as } r_1 \to 0 \text{ \& } r_2 \to \infty} + x\int_{r_1}^{r_2} t^{x-1} e^{-t}\,dt$$

$$\Rightarrow \int_0^\infty t^x e^{-t}\,dt = x\int_0^\infty t^{x-1} e^{-t}\,dt.$$

$$\Rightarrow \Gamma(x+1) = x\,\Gamma(x) \quad \forall\ x > 0.$$ ☒

\# $\therefore\ \forall\ n \in \mathbb{N}$,

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$= n(n-1)\,\Gamma(n-1)$$
$$= \cdots =$$
$$= n\cdot(n-1)\cdots 2\cdot 1\cdot\Gamma(1).$$

Now $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$.

$$\Rightarrow \Gamma(n+1) = n! \quad \forall\ n \geq 1.$$

$\therefore\ \Gamma$ is a cont. analog of factorial fn !!

§ Cauchy principle Value :

Recall : if $f : (-\infty, \infty) \longrightarrow \mathbb{R}$ & if $\exists$ $c \in \mathbb{R}$ s.t.

both I. I's $\int_{-\infty}^{c} f$ & $\int_{c}^{\infty} f$ exist , then

we say $\int_{-\infty}^{\infty} f$ Converges & write

$$\int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f .$$

In this case, RHS is indep. of the choice of c.

$\therefore$ $\int_{-\infty}^{\infty} f$ Converges $\overset{(*)}{\iff}$ $\lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} f$ exists .

Moreover, in this case: $\int_{-\infty}^{\infty} f = \lim_{\substack{R_1 \to \infty \\ R_2 \to \infty}} \int_{-R_1}^{R_2} f$.

[(*) Indeed : $\int_{-\infty}^{\infty} f = \int_{-\infty}^{c} f + \int_{c}^{\infty} f$

$$= \lim_{R_1 \to -\infty} \int_{-R_1}^{c} f + \lim_{R_2 \to \infty} \int_{c}^{R_2} f$$

$$= \lim_{\substack{R_1 \to \infty \\ R_2 \to \infty}} \int_{-R_1}^{R_2} f \quad]$$

Therefore, $\int_{-\infty}^{\infty} f = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} f$ exists & the

limit ~~Converg~~ exists ~~is~~ independently of how R_1 & R_2 approach ∞.

$\therefore$ limit of the fn. $\eta(R_1, R_2) = \int_{-R_1}^{R_2} f$ as $R_1, R_2 \to \infty$.

Clearly, a strong assumption.
& in many cases, this fails to exist.

Instead:

Def: The Cauchy principle value (CPV) of $\int_{-\infty}^{\infty} f$ is defined by:

$$\text{CPV-}\int_{-\infty}^{\infty} f = \lim_{R \to \infty} \int_{-R}^{R} f \quad \text{(if exists)}.$$

Fact: If $\int_{-\infty}^{\infty} f$ exists, then $\text{CPV-}\int_{-\infty}^{\infty} f$ exists &

$$\text{CPV-}\int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f.$$

— HW —. Easy.

The Converse is NOT true:

eg: $\text{CPV}\int_{-\infty}^{\infty} f$ exists but $\int_{-\infty}^{\infty} f$ does not converge, where

$$f(x) = \frac{x}{1+x^2}. \qquad x \in \mathbb{R}.$$

Indeed, $$\int_{-R}^{R} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-R}^{R} \frac{2x}{1+x^2}\,dx \overset{1+x^2=t}{=} \int_{1+R^2}^{1+R^2} \frac{1}{t}\,dt$$

$$= \frac{1}{2}\Big[\log t\Big]_{1+R^2}^{1+R^2} = 0. \qquad \forall\, R>0.$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{x}{1+x^2}\,dx = 0.$$

i.e. $$\text{CPV}-\int_{-\infty}^{\infty} f = 0.$$

~~How~~ However, $$\int_{-\infty}^{\infty} f = \lim_{R\to\infty} \int_{-R}^{0} \frac{x}{1+x^2}\,dx + \lim_{R\to\infty} \int_{0}^{R} \frac{x}{1+x^2}\,dx.$$

$$= \lim_{R\to\infty} \frac{1}{2}\int_{1+R^2}^{1} \frac{1}{t}\,dt$$

$$= \lim_{R\to\infty} \left\{-\frac{1}{2}\log(1+R^2)\right\} = -\infty.$$

$$= \lim_{R\to\infty} \frac{1}{2}\int_{1}^{1+R^2} \frac{1}{t}\,dt = \infty.$$

$\Rightarrow \int_{-\infty}^{\infty} f$ - does not converge.

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§ Sequence & Series of functions.

Recall: A seqn. $\{x_n\} \subseteq \mathbb{R}$ is convergent if $\exists\, x \in \mathbb{R}$ s.t. for $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ $\ni$

$$|x_n - x| < \varepsilon \quad \forall\ n \geq N. \qquad \text{— } (*)$$

$$\Updownarrow$$

For $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ $\ni$

$$|x_n - x_m| < \varepsilon \quad \forall\ n, m \geq N. \quad \Big]\ \text{Cauchy Criterion.}$$

Goal: Replace x_n by f_n $(: S \to \mathbb{R},\ S \subseteq \mathbb{R})$ & talk about closeness, limit, etc.!!

\# **Setting:**

1) $S \subseteq \mathbb{R}$.

2) $\mathcal{F}(S) = \{f : S \to \mathbb{R}\}$ ← A vector space over $\mathbb{R}$. ← Even an algebra.

3) $\{f_n\}_{n=1}^{\infty}$ or simply $\{f_n\}$ will refer a seqn. of fn's $: S \to \mathbb{R}$.

i.e. $\{f_n\} \subseteq \mathcal{F}(S)$.

Goal (again): To talk about " $|f_n - f_m| < \varepsilon$ $\forall\ n, m \geq N$ ".

Q: How to do it?

2 - ways to do it !!

Simply Consider $\{f_n(x)\} \subseteq \mathbb{R}$ $\forall\, x \in S$. Then bring back $|\cdot|$ in $\mathbb{R}$ to the Convergency.

Think a little more: Introduce a suitable concept of distance between 2 functions!! Like "modulus" of functions!!

pointwise

uniform

modulus.

Def: Let $\{f_n\} \subseteq \mathcal{F}(S)$ & $f \in \mathcal{F}(S)$. We say that f_n Converges to f pointwise (or $f_n \xrightarrow{p} f$) if

$$\forall\, x \in S,\ f_n(x) \longrightarrow f(x).$$

\# We also write: $f_n \longrightarrow f$ pointwise.

\# $f_n \xrightarrow{p} f \iff$ For $x \in S$ & $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\, n \geq N$$

DANGER !!

$\boxed{N \equiv N(x, \varepsilon)}$.

\# N depends on ε is acceptable (like $x_n \to x$ Case), but the dependence on x is slightly "less desirable"!!

Will be fixed in the "uniform" part.

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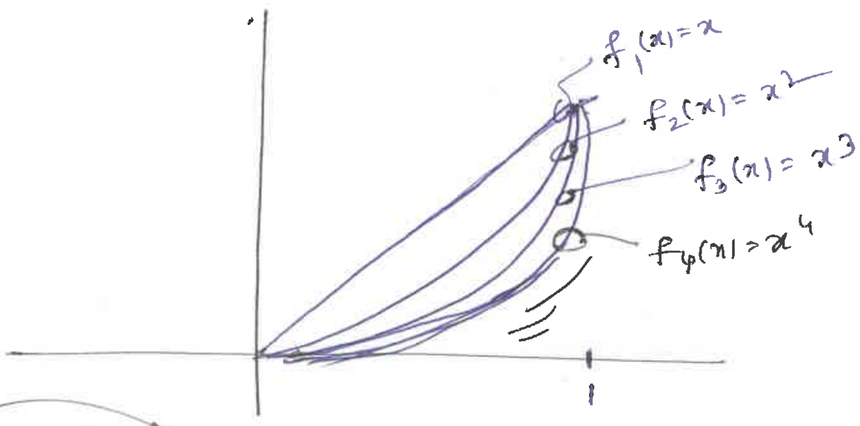
eg:

1) $S = [0,1]$. $\qquad f_n(x) = x^n \quad \forall\ x \in [0,1]$, $n \in \mathbb{N}$.



For each $x \in [0,1)$, we know that $x^n \longrightarrow 0$.

For $x=1$, $\quad x^n \longrightarrow 1$.

$\therefore$ If we define $f : [0,1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases},$$

then $\quad f_n \xrightarrow{p} f$.

[Remark: $\forall\ n \in \mathbb{N}$, $f_n \in C[0,1]$, or even $f_n \in \mathcal{D}[0,1]$. However, the limit (pointwise) f $f \notin C[0,1]$!!

Strange, but reality. We need to fix or identify the trouble!!]

(2) $f_n(x) = \dfrac{1}{x+n} \quad \forall\, n \geq 1,\ x \in [0, \infty).$

$\therefore\ \forall\, x \in [0,\infty),\quad f_n(x) \longrightarrow 0. \qquad \left[\because \frac{1}{x+n} \leq \frac{1}{n} \to 0\right].$

$\Rightarrow\ f_n \xrightarrow{p} f$ where $f(x) = 0 \ \forall\, x \in [0,\infty).$

i.e. $f_n \xrightarrow{p} 0.$

[But something more: For $\varepsilon > 0$, we have:

$$|f_n(x) - 0| = \frac{1}{x+n} \leq \frac{1}{n}.$$

$$\therefore\ |f_n(x) - 0| < \varepsilon \qquad \forall\, n > \frac{1}{\varepsilon}.$$

n does not depend on x!!]

Def: Let $\{f_n\} \subseteq \mathcal{F}(S)$ & $f \in \mathcal{F}(S)$. We say that f_n converges to f uniformly (& write $f_n \xrightarrow{u} f$, or $f_n \longrightarrow f$ unif.) if for $\varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N \ \&\ x \in S. \qquad (*)$$

eg: $f_n \xrightarrow{u} 0$, where $f_n(x) = \dfrac{1}{x+n} \quad \forall\, x \in [0,\infty)$ & $n \in \mathbb{N}$.

Remark: $(*) \iff f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \forall\, n \geq N,\ x \in S.$