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Compatibility :- (Examples) :

eg: Recall $\{f_n\} \subseteq C[0,1]$, where $f_n(x) = x^n$; $x \in [0,1]$, $n \in \mathbb{N}$.

Then $f_n \xrightarrow{p} f$, where $f(x) = \begin{cases} 0 & x \in [0,1). \\ 1 & x = 1. \end{cases}$

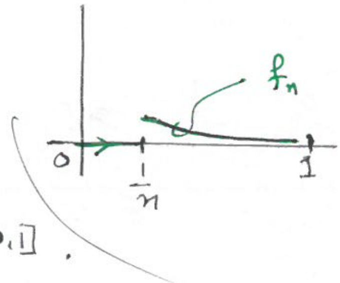
Clearly, $f \notin C[0,1]$.

$\therefore$ ~~$C[0,1]$~~ pointwise convergency & continuity are not compatible.

[Q: If $\{f_n\}$ are cont. & $f_n \xrightarrow{u} f$ on S
$\overset{?}{\Rightarrow}$ f is cont ?] Also, if f_n is cont. at x_0, does $\lim\limits_{n\to\infty} \lim\limits_{x\to x_0} f_n = \lim\limits_{x\to x_0} \lim\limits_{n\to\infty} f_n \left(= \lim\limits_{x\to x_0} f\right)$?]

eg: $f_n(x) = \begin{cases} 0 & x \in [0, 1/n] \\ \frac{1}{x} & x \in [0,1] \setminus [0, 1/n] \end{cases}$.

$\therefore \{f_n\} \subseteq \mathcal{B}[0,1]$.



However, $\lim\limits_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \in (0,1] \end{cases}$.

$\Rightarrow \lim\limits_{n\to\infty} f_n(x) := f(x) \quad \forall\ x \in [0,1]$

is NOT bdd.

$\therefore$ pointwise convergency does not preserve boundedness !!

Q: What about u.c ?

eg: Consider an enumeration $\{r_n\}_{n=1}^{\infty}$ of rationals $\mathbb{Q} \cap [0,1]$.

Define $f_n(x) = \begin{cases} 0 & \text{if } x = r_1, \dots, r_n \\ 1 & \text{if } x \in [0,1] \setminus \{r_1, \dots, r_n\}. \end{cases}$

$\therefore f_n \in \mathcal{R}[0,1] \quad \forall n.$ [$\because f_n$ is discont. at finitely many points. $\forall n$.]

Now for $m \in \mathbb{N}$, we know: $f_n(r_m) = 0 \quad \forall n \geqslant m$.

$\Rightarrow f_n(r_m) \longrightarrow 0$ as $n \to \infty$. $\forall m \in \mathbb{N}$.

$\therefore \forall x \in \{r_n\}_{n=1}^{\infty}$, $\lim_{n \to \infty} f_n(x) = 0$.

Next, let $x \in [0,1] \setminus \mathbb{Q}$.

$\therefore f_n(x) = 1 \ \forall n \Rightarrow \lim_{n \to \infty} f_n(x) = 1$.

$\therefore f_n \xrightarrow{p} f$ on $[0,1]$,

where $f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c. \end{cases}$

But we know that $f \notin \mathcal{R}[0,1]$.

$\therefore \lim_{n \to \infty} f_n(x) \notin \mathcal{R}[0,1]$. $\therefore \mathcal{R}[a,b]$ is Not closed under pointwise convergency!!

[Q: What if $f_n \to f$ unif. ?]

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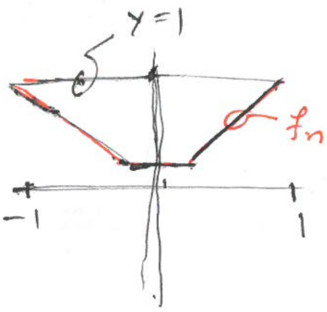
eg: Recall that $f_n \to f$ uniformly on $[-1,1]$, where

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq \frac{1}{n}. \\ |x| & \text{if } \frac{1}{n} < |x| \leq 1. \end{cases}$$



& $f(x) = |x|$, $x \in [-1,1]$.

Note that f_n is diff. at 0 $\forall n$.

However, f is NOT diff. at 0.

Here the situation is even worse : as $f_n \to f$ unif. on $[-1,1]$.

$\Downarrow$

[U.C is not compatible with diff!!]

All the examples yield negative feeling about the following compatibility issue :

Fact

Suppose $\{f_n\} \subseteq \mathcal{F}(S)$, $f \in \mathcal{F}(S)$. Suppose $f_n \to f$ $\boxed{\text{pointwise}}$ on S.

Let f_n is cont. on S $\forall n$. $\overset{?}{\Longrightarrow}$ f is cont. on S ? NO!

Let $f_n \in \mathcal{B}(S)$ $\forall n$. $\overset{?}{\Longrightarrow}$ $f \in \mathcal{B}(S)$? NO!

Let $f_n \in \mathcal{R}[a,b]$. $\forall n$. $\overset{?}{\Longrightarrow}$ $f \in \mathcal{R}[a,b]$? NO!

If So, then must it be true that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n \quad \left(= \int_a^b f \right) \quad ? \quad \leftarrow ??$$

(pending)

Let f_n is diff. at $x \in S$, $\forall n$. $\overset{?}{\Longrightarrow}$ f' exists at x ? NO!

If So, then must it be true that

$$\lim_{n\to\infty} f_n'(x) = f'(x) \quad ? \quad \leftarrow ??$$

(pending)

\# Suppose $\lim_{x \to x_0} f_n$ exists $\forall\, n$. $\overset{?}{\Rightarrow}$ $\lim_{x \to x_0} f$ exist? NO!

If so, is it true that

$$\lim_{n \to \infty} \left(\lim_{x \to x_0} f_n \right) = \lim_{x \to x_0} f \quad ?$$

~~Pending~~ NO

eg:
See Page-13

eg: $f_n(x) = \begin{cases} 0 & 0 < x \le 1/n \\ \frac{1}{x} & 1/n < x \le 1 \end{cases}$

$\Rightarrow \lim_{x \to 0} f_n$ exists $\forall\, n$. But $\lim_{n \to 0} \left(\lim f_n \right)$ DNE !!

eg: Consider $f_n(x) = x^n$, $x \in (0,1)$.

We know, $\lim_{n \to \infty} f_n(x) = f(x)$ $\forall\, x$, where $f \equiv 0$.

i.e. $f_n \to 0$ pointwise on $(0,1)$.

Now, $\lim_{x \to 1} f_n(x) = 1$ $\forall\, n$.

But $\lim_{x \to 1} f(x) = 0$.

$$\therefore \lim_{n \to \infty} \lim_{x \to 1} f_n(x) \neq \lim_{x \to 1} \lim_{n \to \infty} f_n(x).$$

— x —

All in all:

Pointwise convergence is a natural concept $\boxed{\text{but}}$ with a number of disadvantages !!

AND, Indeed, one would like to capture all the above properties of convergence !!

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In the following, we prove that with uniform convergency, all problems disappear.

BUT, NOT ~~with~~ for differentiability !! ⟵ We will work this out too!!

Thm: Let $x_0 \in S$ & $f_n \xrightarrow{u} f$ on $S \setminus \{x_0\}$. If $\lim\limits_{x \to x_0} f_n$ exists $\forall n$, then $\lim\limits_{x \to x_0} f$ also exists. In this case,

$$\lim_{n \to \infty} \lim_{x \to x_0} f_n = \lim_{x \to x_0} f.$$

[i.e. $\lim\limits_{n \to \infty} \lim\limits_{x \to x_0} f_n = \lim\limits_{x \to x_0} \lim\limits_{n \to \infty} f_n$.]

∴ Interchange of limits.

U.C & limit

Proof: Let $\varepsilon > 0$. Since $f_n \xrightarrow{u} f$, by Cauchy criterion, $\exists N \in \mathbb{N}$ s.t.

$$\|f_n - f_m\| < \varepsilon/2 \quad \forall n, m \geq N. \text{ ON } S \setminus \{x_0\} \quad \text{—— } (*)$$

$\forall n \in \mathbb{N}$, set $a_n := \lim\limits_{x \to x_0} f_n$.

Now, $a_n - a_m = \lim\limits_{x \to x_0} [f_n(x) - f_m(x)]$.

$$\Rightarrow |a_n - a_m| = \lim_{x \to x_0} |f_n(x) - f_m(x)| \quad \left[\because ||a|-|b|| \leq |a-b|\right]$$

$$\leq \frac{\varepsilon}{2} \quad [\text{by } (*)] \quad \forall m, n \geq N$$

$\Rightarrow |a_n - a_m| \leq \varepsilon/2 \quad \forall m, n \geq N$.

$\Rightarrow \{a_n\}$ is Cauchy.

$\therefore \exists a \in \mathbb{R}$ s.t. $a := \lim\limits_{n \to \infty} a_n$.

Again, $f_n \xrightarrow{u} f$ on $S \setminus \{x_0\}$ gives: $\exists\, n_0 \in \mathbb{N}$ s.t.

$$\| f_n - f \| < \varepsilon/3 \quad \forall\, n \geq n_0, \quad \text{on } S \setminus \{x_0\}. \qquad \text{(i)}$$

Also, for $a_n \to a$, $\exists\, \tilde{n}_0 \in \mathbb{N}$ s.t.

$$|a_n - a| < \varepsilon/3 \quad \forall\, n \geq \tilde{n}_0. \qquad \text{(ii)}$$

Set $\hat{n} := \max\{n_0, \tilde{n}_0\}$. Focus is on $\hat{n}$ now!!

$\because \lim_{x \to x_0} f_{\hat{n}} = a_{\hat{n}}$, $\exists\, \delta > 0$ s.t.

$$|f_{\hat{n}}(x) - a_{\hat{n}}| < \varepsilon/3 \quad \forall\, x \in S \setminus \{x_0\} \text{ s.t. } |x - x_0| < \delta. \qquad \text{(iii)}$$

$\therefore$ for each $x \in S \setminus \{x_0\}$ & $|x - x_0| < \delta$, we have:

$$\begin{aligned} |f(x) - a| &\leq |f(x) - f_{\hat{n}}(x)| + |f_{\hat{n}}(x) - a_{\hat{n}}| \\ &\quad + |a_{\hat{n}} - a| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \text{(by (i)–(iii))} \\ &= \varepsilon. \end{aligned}$$

Typical $\varepsilon/3$-argument.

$\therefore |f(x) - a| < \varepsilon \quad \forall\, x \in S \setminus \{x_0\}$ & $|x - x_0| < \delta$.

$\Rightarrow \lim_{x \to x_0} f = a$.

i.e. $\lim_{x \to x_0} f = \lim_{n \to \infty} \lim_{x \to x_0} f_n$.

$\boxed{//}$

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unif. Cont. & limit.

Thm: (Continuity) Let $f_n \xrightarrow{u} f$ on S. Let $x_0 \in S$ & let each f_n is continuous at x_0. Then f is also cont. at x_0.

Proof: We know $\lim\limits_{x \to x_0} f_n = f_n(x_0) \quad \forall n$. [$\because f_n$ is cont. at x_0].

Also, $f_n \xrightarrow{u} f \Rightarrow \lim\limits_{n \to \infty} f_n(x_0) = f(x_0)$.

$$\therefore f(x_0) = \lim_{n \to \infty} f_n(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n$$

$$= \lim_{x \to x_0} \lim_{n \to \infty} f_n \qquad \text{[by previous thm]}$$

$$= \lim_{x \to x_0} f$$

$\Rightarrow$ f is cont. at x_0. ☐

Thm: (Bounded fn's). Let $\{f_n\} \subseteq \mathcal{B}(S)$ & $f_n \xrightarrow{u} f$ on S.

Then $f \in \mathcal{B}(S)$.

u.c & bdd

[$\because$ $\mathcal{B}(S)$ is closed under uniform limits.]

Proof: $\because f_n \xrightarrow{u} f$ on S, for $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t.

$$\|f_n - f\| < 1 \quad \forall n \geq N.$$

Then, $\forall x \in S$, we have:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$$

$$< 1 + \|f_N\|.$$

$$\Rightarrow \|f\| < 1 + \|f_N\| \Rightarrow f \in \mathcal{B}(S).$$

☐

Thm: (Riemann integration) Let $\{f_n\} \subseteq \mathcal{R}[a,b]$ & let $f_n \xrightarrow{u} f$ on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ & $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

u.c & integ.

$$\left[\text{i.e. } \lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n \right].$$

Interchange of limits.

Proof: Let $\varepsilon > 0$. Since $f_n \xrightarrow{u} f$ on $[a,b]$, $\exists\ N \in \mathbb{N}$ s.t.

$$\|f_n - f\| < \frac{\varepsilon}{b-a}. \quad \forall\ n \geq N. \qquad \text{(i)}$$

Focus on N.

In particular: $\|f_N - f\| < \frac{\varepsilon}{b-a}$.

$$\text{i.e. } f(x) - \frac{\varepsilon}{b-a} < f_N(x) < f(x) + \frac{\varepsilon}{b-a}. \qquad \text{(ii)}$$

$\because f_N \in \mathcal{R}[a,b]$, $\exists\ P \in \mathcal{P}[a,b]$ s.t.

$$U(f_N, P) - L(f_N, P) < \varepsilon. \qquad \text{(iii)}$$

Now (ii) $\Rightarrow$ $f(x) < f_N(x) + \frac{\varepsilon}{b-a} \quad \forall\ x \in [a,b]$

$\Rightarrow U(f, P) < U(f_N, P) + \varepsilon$.

|||ly (ii) $\Rightarrow$ $f(x) > f_N(x) - \frac{\varepsilon}{b-a} \quad \forall\ x \in [a,b]$

$\Rightarrow L(f, P) > L(f_N, P) - \varepsilon$.

So, (iii) $\Rightarrow$ $U(f_N, P) - L(f_N, P) < \varepsilon$

$\Rightarrow \big(U(f,P) - \varepsilon\big) - \big(L(f,P) + \varepsilon\big) < \varepsilon$.

$\Rightarrow U(f,P) - L(f,P) < 3\varepsilon$.

$\Rightarrow f \in \mathcal{R}[a,b]$.

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Finally, we prove that $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

$\forall\, x \in [a,b]$, we have

$$\left| \int_a^x f - \int_a^x f_n \right| = \left| \int_a^x (f_n - f) \right| \leq \int_a^x |f_n - f| .$$

$\therefore$ By (i), $\forall\, n \geq N$, we have:

$$\left| \int_a^x f - \int_a^x f_n \right| \leq \frac{\varepsilon}{b-a} \times \int_a^x dx = \frac{\varepsilon}{b-a} \times (x-a)$$

$$\leq \varepsilon \qquad \forall\, x \in [a,b].$$

$\therefore$ $\left\{ \int_a^x f_n \right\}$ converges unif. to $\int_a^x f$ on $[a,b]$.

In particular: $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. $\boxed{\checkmark}$

In fact: we proved that:

$$\lim_{n\to\infty} \int_a^x f_n = \int_a^x \lim f_n \qquad \forall\, x \in [a,b].$$