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$$\therefore \quad |S_n(x) - S_m(x)| \leq \frac{2|x|^m}{|1-x|} \leq \frac{2|x|^m}{1-|x|}$$

$$\leq \frac{2\varepsilon^m}{1-\varepsilon}. \qquad \left(\because\ x \in [-\varepsilon, \varepsilon] \ \& \ 0<\varepsilon<1.\right)$$

i.e. $\quad |S_n(x) - S_m(x)| \leq 2 \times \dfrac{\varepsilon^m}{1-\varepsilon} \quad \forall\ x \in [-\varepsilon, \varepsilon]$ & $n > m$.

$$\Longleftrightarrow \left\| \sum_{k=m+1}^{n} f_k \right\| \leq 2\,\frac{\varepsilon^m}{1-\varepsilon} \quad \forall\ n > m.$$

$\because$ $0<\varepsilon<1$

by Cauchy Criterion

$\left[\because\ \varepsilon^m \to 0 \text{ as } m \to \infty,\ \text{for } \tilde{\varepsilon} > 0\ \exists\ N \in \mathbb{N}\right.$

s.t. $\quad 2\,\dfrac{\varepsilon^m}{1-\varepsilon} < \tilde{\varepsilon} \quad \forall\ m \geq N.$

$\therefore\ \forall\ n > m \geq N$

$$\left.\left\| \sum_{k=m+1}^{n} f_k \right\| < \tilde{\varepsilon}\ \right]$$

$$\Rightarrow \sum_{n=0}^{\infty} x^n \text{ is u.c. on } [-\varepsilon, \varepsilon] \quad \forall\ 0<\varepsilon<1.$$

$\square$

\# Of Course: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall\ x \in [-\varepsilon, \varepsilon].$

\# There are other ways to prove the above conclusion!

— x —

M-test

Thm: Let $\{f_n\} \subseteq \mathcal{F}(S)$, $\|f_n\| \leq M_n \ \forall\ n \geq 1$ & Suppose $\displaystyle\sum_{n=1}^{\infty} M_n < \infty$. Then $\sum f_n$ converges uniformly & absolutely on S.

[**Def:** $\sum f_n$ is absolutely convergent if $\sum |f_n(x)|$ converges $\forall x$.]

Proof: $\because \|f_n\| < M_n \Rightarrow |f_n(x)| < M_n \; \forall x \,\&\, n$,

by comparison test, $\sum f_n$ is absolutely convergent.

Now, $\forall n > m$, we have

$$\| S_n - S_m \| = \left\| \sum_{k=m+1}^{n} f_k \right\| \leq \sum_{k=m+1}^{n} \|f_k\| \leqslant \sum_{k=m+1}^{n} M_k .$$

$\because \sum_{n=1}^{\infty} M_n < \infty$, by Cauchy criterion, $\sum f_n$ is u.c. ☒

eg: ① $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ $\quad p > 1$, $x \in \mathbb{R}$.

Here $f_n(x) := \frac{\sin nx}{n^p}$ $\forall n$, $x \in \mathbb{R}$.

$\therefore \|f_n\| \leq \frac{1}{n^p}$.

$\because \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $(p>1)$, by Weierstrass M-test,

$\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ converges absolutely & uniformly on $\mathbb{R}$.

② Set $f_n(x) = \frac{x}{n+n^2x^2}$ $(x \in \mathbb{R})$. $\therefore f_n(0) = 0 \; \forall n$.

For $x \neq 0$; $|f_n(x)| = \frac{|x|}{n+n^2x^2} = \frac{1}{\underbrace{\frac{n}{|x|} + n^2|x|}_{\geqslant 2n^{3/2}}}$

$$\leqslant \frac{1}{2n^{3/2}} .$$

$\therefore$ By Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{x}{n+n^2x^2}$ is absolutely & u.c. on $\mathbb{R}$.

☒

(3) **u.c but NOT absolutely Convergent:** (Not so easy example. But we also have almost a trivial one: wait).

useful technique.

Consider $\sum f_n$ on $\mathbb{R}$, with

$$f_n(x) = \frac{(-1)^{n+1}}{n+x^2} \quad \forall\ x \in \mathbb{R}.$$

i.e. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+x^2}$ on $\mathbb{R}$.

Now for each fixed $x \in \mathbb{R}$, $\displaystyle |f_n(x)| = \frac{1}{n+x^2}$.

But $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+x^2}$ is divergent. [Note: For $x \in \mathbb{R}$ fixed, given $n \in \mathbb{N}$ $\exists\ m \in \mathbb{N}$ s.t. $n+x^2 < n+m$ $\Rightarrow \frac{1}{n+x^2} > \frac{1}{n+m}$]

$\Rightarrow$ $\sum f_n$ is not absolutely convergent $\forall\ x \in \mathbb{R}$.

Now we prove that $\sum f_n$ is u.c.

$\forall\ n \in \mathbb{N}$, observe that:

$$S_{2n}(x) = \left(\frac{1}{1+x^2} - \frac{1}{2+x^2}\right) + \left(\frac{1}{3+x^2} - \frac{1}{4+x^2}\right) + \cdots + \left(\frac{1}{2n-1+x^2} - \frac{1}{2n+x^2}\right) \quad \forall\ x \in \mathbb{R}.$$

$\because$ all terms are >0, it follows that $S_{2n}(x) \uparrow$.

Also, $\sum f_n(x)$ is an an alternating series & easy to see it converges $\forall\ x \in \mathbb{R}$. ← Why?

Set $f(x) := \sum f_n(x) \quad \forall\ x \in \mathbb{R}$.

$\therefore\ f(x) - S_{2n}(x) > 0 \quad \forall\ x$.

Also, $\displaystyle f(x) - S_{2n}(x) = \frac{1}{2n+1+x^2} - (+\text{ve no.})$

$$< \frac{1}{2n+1+x^2} < \frac{1}{2n}$$

$\forall\ x \in \mathbb{R}$.

i.e. $f(x) - S_{2n}(x) < \frac{1}{2n}$ $\forall\, x \in \mathbb{R},\ n \in \mathbb{N}$.

$||^{ly}$ $\quad$ $S_{2n+1}(x) - f(x) > 0$

& $S_{2n+1}(x) - f(x) < \frac{1}{2n}$.

$\therefore$ $\forall\, n \in \mathbb{N}$ & $x \in \mathbb{R}$,

$$\left.\begin{aligned} 0 &< f(x) - S_{2n}(x) < \frac{1}{2n} \\ \& \quad 0 &< S_{2n+1}(x) - f(x) < \frac{1}{2n} \end{aligned}\right\} \Downarrow$$

i.e. $|S_n(x) - f(x)| < \frac{1}{2n}$ $\forall\, n$ & $x \in \mathbb{R}$.

$\Rightarrow \quad S_n \xrightarrow{u} f$ on $\mathbb{R}$

$\therefore \sum_{n=1}^{\infty} f_n$ is u.c. on $\mathbb{R}$. $\square$

Even Simpler: $f_n(x) \equiv \frac{(-1)^{n+1}}{n}$ $\forall\, n, x$. Then $\sum f_n$ is u.c. but NOT A.c.!!

Thm: Suppose $\sum f_n = f$ uniformly on $S \setminus \{x_0\}$ for some $x_0 \in S$.

Limit - Series thm.

If $\lim_{x \to x_0} f_n$ exists $\forall\, n \in \mathbb{N}$, then

$\sum_{n=1}^{\infty} \lim_{x \to x_0} f_n$ ~~exist~~ Converges &

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n.$$

Interchange of limits.

Proof: For $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $\left\| \sum_{k=m+1}^{n} f_k \right\| < \varepsilon/2$ i.e.

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon/2 \quad \forall\, n > m \geq N \ \& \ x \in S \setminus \{x_0\}$$

$\because \lim_{x \to x_0} f_k$ exists $\forall\, k$, & as the above sum is finite, it follows that

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$$\left| \sum_{k=m+1}^{n} \lim_{x\to x_0} f_k \right| < \varepsilon \quad \forall \; n > m \geq N.$$

$$\Rightarrow \quad \alpha := \sum_{n=1}^{\infty} \lim_{x\to x_0} f_n \quad \text{Converges.}$$

You can start the proof from right HERE!

Finally, since $\sum f_n = f$ unif. on $S \setminus \{x_0\}$,

$$S_n \longrightarrow f \quad \text{uniformly on } S\setminus\{x_0\}.$$

$\because$ $S_n = \sum_{k=1}^{n} f_n$ & $\lim_{x\to x_0} f_n$ exists $\forall\, n$, ~~we have~~

the limit $\lim_{x\to x_0} S_n = \sum_{k=1}^{n} \lim_{x\to x_0} f_k$ exists $\forall n$.

$\therefore$ By "limit – u.c thm", it follows that

$$\lim_{x\to x_0} S_n \longrightarrow \lim_{x\to x_0} f$$

i.e. $$\lim_{n\to\infty} \left(\lim_{x\to x_0} \sum_{k=1}^{n} f_k \right) = \lim_{x\to x_0} f$$

i.e. $$\cancel{\lim_{n\to\infty}} \sum_{n=1}^{\infty} \lim_{x\to x_0} f_n = \lim_{x\to x_0} \sum f_n .$$

$\boxtimes$

llly using partial sums & corresponding results in u.c. we have the following.

\# Let $\sum f_n = f$ unif. on S. If f_n is bdd $\forall n$, then f is also bdd.

\# Let $\sum f_n = f$ unif. on $[a,b]$, $f_n \in \mathcal{R}[a,b]$ $\forall n$.

Then $f \in \mathcal{R}[a,b]$ & $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$.

\# If $f_n \in C(S), \forall n$, & $\sum f_n$ converges unif. then

$$\sum f_n \in C(S).$$

\# $|||^{ly}$ derivatives.

Similar proof.

$|||^{ly}$

~~**Thm:** (Dini's thm on u.c) Let $S \subseteq \mathbb{R}$ be compact, $\{f_n\} \subseteq C(S)$ & $f_n \to f \in C(S)$ pointwise. If $\{f_n\}$ is monotonic~~~~ally decreasing~~ (i.e. $\{f_n(x)\} \downarrow$ or $\uparrow$ $\forall x \in S$), then

$$f_n \longrightarrow f \text{ uniformly on } S.$$

~~Proof: If possible, let $f_n \not\to f$ unif. on S.~~

~~$\therefore \exists \ \varepsilon > 0$ s.t. $\underbrace{\|f_n - f\|}_{} > \varepsilon$ for infinitely many $n \in \mathbb{N}$.~~

$$\| \quad \sup_{x \in S} \{ |f_n(x) - f(x)| \}.$$

$$\| \quad \sup_{x \in S} \{ f_n(x) - f(x) \}$$

$$[\because f_n(x) \downarrow f(x) \ \forall x \in S].$$

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Thm: (Dini's theorem on u.c.).

Let $S \subseteq \mathbb{R}$ be compact, $\{f_n\} \subseteq C(S)$ & let $f_n \longrightarrow f \in C(S)$ pointwise. If $\{f_n\}$ is monotonic (i.e. $\{f_n(x)\} \downarrow$ or $\uparrow$ $\forall x \in S$) then $f_n \longrightarrow f$ uniformly on S.

For series of f_n's; we know if $\sum f_n = f$ unif. & f_n cont., then f is cont. This is a "kind" of converse.

Proof: WLOG: assume $f_n \downarrow$ i.e.

$$f_n(x) \geqslant f_{n+1}(x) \quad \forall x \in S, n \geqslant 1.$$

Set $F_n = f_n - f \quad \forall n.$ $\longleftarrow$ $(\because f_n \xrightarrow{p} f)$.

$\therefore \{F_n\} \subseteq C(S)$, $F_n \downarrow$, $F_n \geqslant 0$.

i.e. $\quad F_n(x) \geqslant F_{n+1}(x) \geqslant 0 \quad \forall x \in S, n \geqslant 1.$

Recall $\sup\{F_n(x) : x \in S\} = \|F_n\|$, $\forall n$.

Claim: $\|F_n\| \longrightarrow 0$.

Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, define

$$O_n := F_n^{-1}(-\infty, \varepsilon) \quad \left(= \{x \in S : F_n(x) < \varepsilon\} \right)$$

$\because F_n \in C(S)$, we have that O_n open in S $\forall n$.

Also $F_n \downarrow \implies O_{n+1} \supseteq O_n \quad \forall n.$

$\because f_n(x) \longrightarrow f(x) \ \forall x \in S$, it follows that

$$F_n(x) \longrightarrow 0 \quad \forall x \in S.$$

$\therefore$ For each $x \in S$, $\exists N_x \in \mathbb{N}$ s.t.

$$F_{N_x}(x) < \varepsilon.$$

$\implies x \in O_{N_x}$.

$\therefore$ For $x \in S$, $\exists N \in \mathbb{N}$ s.t. $x \in O_N$.

$\implies \bigcup_{n=1}^{\infty} O_n = S$. $\longleftarrow$ $\therefore \{O_n\}$ an open cover of S.

But S is compact.

$\Rightarrow \exists\, N \in \mathbb{N}$ s.t. $\displaystyle\bigcup_{n=1}^{N} O_n = S$.

i.e. $O_N = S$. $\leftarrow \because O_n \uparrow$.

i.e. $\{x \in S : F_N(x) < \varepsilon\} = S$.

$\Rightarrow F_N(x) < \varepsilon \quad \forall\, x \in S$.

$\Rightarrow \|F_N\| < \varepsilon$.

But $F_n \downarrow \Rightarrow \|F_n\| \longrightarrow 0$. ▨

$(\because \|F_n\| \leq \|F_N\| < \varepsilon \;\forall\, n \geq N)$.

Remark:

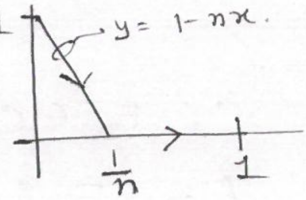
① S ~~is~~ compact is necessary:

$f_n(x) = x^n$ on $(0,1)$.

$\therefore f_n \downarrow$ & $f_n \xrightarrow{p} 0$. But $f_n \not\to 0$ unif. on $(0,1)$.

② Continuity of f ($= \lim f_n$, pointwise) ~~is continuous~~ is also necessary:

$$f_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq 1/n \\ 0 & 1/n < x \leq 1. \end{cases}$$



$\frac{x}{1/n} + \frac{y}{1} = 1$

$\Rightarrow y = 1 - nx$

That's the idea.

$\therefore f_n \xrightarrow{p} f$

where $f(x) = \begin{cases} 1 & x = 0. \\ 0 & x \in (0,1]. \end{cases}$

$\therefore f \notin C[0,1]$.

& $f_n \not\to f$ unif. as $\|f_n - f\| = 1 \;\forall\, n$.

$\Rightarrow \|f_n - f\| \not\to 0$. ▨

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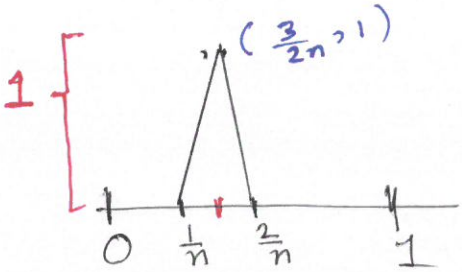
(3) monotonicity of f_n ~~monotonic~~ is also necessary :

Define $f_n : [0,1] \longrightarrow \mathbb{R}$ by



$\therefore$ $f_n \in C[0,1]$ & f_n not monotone.

Also $f_n \longrightarrow 0$ pointwise but

$$\|f_n\| = 1 \ \forall n \implies f_n \not\to 0 \text{ unif.}$$

$\boxtimes$

— x —.