Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Home Assignment I — Due Date : 13 February 2022
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(1) **Vandermonde matrix:** Fix $n \geq 2$. Let $x_1, x_2, \ldots, x_n$ be real numbers. Then the associated Vandermonde matrix is defined as:

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \ldots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \ldots & x_m^{n-1} \end{bmatrix}.$$

(i) Show that $\det(V) = \Pi_{1 \leq i < j \leq n}(x_j - x_i)$. (ii) Show that V is non-singular if and only if x_j's are distinct.

(2) **Companion matrix:** Given a monic polynomial

$$p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} + t^n,$$

its companion matrix is defined as the matrix:

$$C = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -c_0 & -c_1 & -c_2 & \ldots & -c_{n-1} \end{bmatrix}.$$

(i) Compute the characteristic polynomial $p_C(t) = \det(tI - C)$. (ii) If a is a root of the polynomial p, show that

$$\begin{pmatrix} 1 \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{pmatrix}$$

is an eigenvector of C with eigenvalue a.

(3) Prove $\det(AB) = \det(A).\det(B)$ using Leibniz formula for determinants.

(4) Prove that transpose of a permutation matrix is its inverse.

(5) Compute the inverse of

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 4 & 2 & 5 \end{bmatrix}$$

in two different ways by considering it as composed of block matrices of sizes 2×2, 2×2 or 3×3 and 1×1 on the diagonal and using the formulae for inverses of block matrices.

(6) For any real matrix A, show that $A^t A = 0$ if and only if $A = 0$.

(7) Suppose A is a non-singular matrix and u, v are vectors such that $(A + uv^t)$ is also non-singular (Here v^t denotes the transpose of the column vector v). Show that

$$(A + uv^t)^{-1} = A^{-1} + \frac{(A^{-1}u)(v^t A^{-1})}{1 + v^t A^{-1} u}.$$

(8) For any square matrix, the **trace** is defined as the sum of diagonal entries. That is, trace $(A) = \sum_{i=1}^{n} a_{ii}$ for $A = [a_{ij}]_{1\leq i,j\leq n}$. Show that
  (i) trace $(A+B) =$ trace $(A)+$ trace (B), for any A, B.
  (ii) trace $(AB) =$ trace (BA), for any A, B.
  (iii) trace $(M^{-1}AM) =$ trace (A), if M is invertible.

(9) (Weinstein - Aronszajn identity) Let A, B be $m \times n$ and $n \times m$ matrices. Show that

$$\det(I_m + AB) = \det(I_n + BA).$$

Here I_m, I_n denote the identity matrices of respective sizes. (Hint: Try computing the determinant of the block matrix

$$M := \begin{bmatrix} I_m & -A \\ B & I_n \end{bmatrix}$$

in two different ways.)

(10) **A trick question:** Suppose A, B are matrices such that $A + B = AB$. Show that $B + A = BA$.