Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Home Assignment II | Due Date : 20 March 2022
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Remark: Standard inner product is considered on $\mathbb{R}^n$ and $\mathbb{C}^n$ unless some other inner product is explicitly mentioned.

(1) Define 'l^1-norm' on $\mathbb{C}^n$ by
$$\|x\|_1 = \sum_{j=1}^{n} |x_j|, \ \forall x \in \mathbb{C}^n.$$
Show that (i) $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1, \ \forall x, y \in \mathbb{C}^n$. (ii) $\|x\|_1 = 0$ if and only if $x = 0$. (iii) $\|ax\|_1 = |a|\|x\|_1$ for all $a \in \mathbb{C}, x \in \mathbb{C}^n$. (iv) For $n \geq 2$ there is no inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ such that
$$\|x\|_1 = (\langle x, x \rangle)^{\frac{1}{2}}, \ \forall x \in \mathbb{C}^n.$$

(2) Let V be a finite dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $B : V \to V$ be an invertible linear map. Show that
$$\langle x, y \rangle_B := \langle Bx, By \rangle, \forall x, y \in V,$$
defines an inner product on V.

(3) (i) Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map. Show that $\langle v, Av \rangle = 0$ for all $v \in \mathbb{C}^n$ implies $A = 0$. (ii) Show that in general the result in (i) is not true if $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$. (Hint: Get a counter example with $n = 2$.)

(4) Let V, W be finite dimensional inner product spaces and let $A : V \to W$ be a linear map. Show that
$$\ker\ (A) = (\text{range}\ (A^*))^{\perp}.$$
(Here 'ker' stands for kernel.)

(5) Let S be a non-empty subset of a finite dimensional inner product space V.
(i) Show that
$$(S^{\perp})^{\perp} = \ \text{span}\ (S).$$
(ii) Show that $((S^{\perp})^{\perp})^{\perp} = S^{\perp}$.

(6) Let U be an $n \times n$ unitary matrix. Define $D = [d_{ij}]_{1 \leq i,j \leq n}$ by
$$d_{ij} = |u_{ij}|^2, \ 1 \leq i, j \leq n.$$
Show that D is a doubly stochastic matrix. Such matrices are known as 'unitary stochastic' matrices. Show that not every doubly stochastic matrix is a unitary stochastic matrix.

(7) Let λ be an eigen value of a unitary matrix. Show that $|\lambda| = 1$.

(8) Suppose $\{a_1, a_2, \ldots, a_n\}$ are eigenvalues of a matrix A. Suppose $\{b_1, \ldots, b_n\}$ are eigenvalues of a matrix B. Show that in general eigenvalues of $A + B$ are not given by $\{a_1 + b_1, \ldots, a_n + b_n\}$. However, this the case if $B = bI$ for some $b \in \mathbb{C}$.

(9) Let V_0 be the subspace

$$V_0 = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 = 0 \}$$

of $\mathbb{C}^3$. Write down the matrix of the projection map on to V_0 in standard basis.

(10) Suppose $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ are two ortho-normal bases of $\mathbb{C}^n$. Then $\mathcal{B}$ and $\mathcal{C}$ are said to be **mutually unbiased** if

$$|\langle b_i, c_j \rangle| = \gamma, \ \forall 1 \leq i, j \leq n.$$

for some fixed $\gamma \in \mathbb{C}$. (i) Show that if $\mathcal{B}$ and $\mathcal{C}$ are mutually unbiased orthonormal bases and γ is above, then $\gamma = \frac{1}{\sqrt{d}}$. (ii) Obtain three mutually unbiased orthonormal bases $\mathcal{B}, \mathcal{C}, \mathcal{D}$ for $\mathbb{C}^2$ (any two of them should be mutually unbiased).

(Hint: You may take $\mathcal{B} = \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$ and $\mathcal{C} = \{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \}$.

(11) **Challenge Problem (Optional):** Get seven mutually unbiased bases for $\mathbb{C}^6$ or show that it is not possible to get that many.