Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Home Assignment III — Due Date : 11 April 2022
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Remark: Standard inner product is considered on $\mathbb{R}^n$ and $\mathbb{C}^n$ unless some other inner product is explicitly mentioned.

(1) A matrix A is said to be **nilpotent** if $A^k = 0$ for some $k \in \mathbb{N}$. Show that if a normal matrix A is nilpotent then $A = 0$.
(2) A matrix S is said to be **skew-hermitian** if $S^* = -S$. Show that S is skew-hermitian iff iS is self-adjoint. Show that S is skew-hermitian iff S is normal and all its eigenvalues are purely imaginary, that is, they are of the form it where $t \in \mathbb{R}$.
(3) Show that a matrix is normal if and only if its real and imaginary parts commute.
(4) Show that sum of two normal matrices need not be normal. Show that product of two normal matrices need not be normal.
(5) Write down the spectral decomposition of the following matrices:

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(6) Let A be a self-adjoint matrix. Show that there exist two positive matrices A_+, A_- such that

$$A = A_+ - A_-, \ A_+.A_- = A_-A_+ = 0.$$

(Hint: Use spectral theorem)
(7) Suppose $d_1, d_2, \ldots, d_n$ are n-complex numbers and $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation. Show that diagonal matrices D and E with diagonal entries:

$$d_{ii} = d_i, \ e_{ii} = d_{\sigma(i)}, \ 1 \leq i \leq n$$

are unitarily equivalent. Show that if two normal matrices A, B are similar then they are unitarily equivalent.
(8) If A is an $m \times m$ matrix and B is an $n \times n$ matrix, their **direct sum** is defined as the block matrix

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Show that $A \oplus B$ is normal (respectively unitary, self-adjoint, projection, positive) if and only if both A and B are normal (respectively unitary, self-adjoint, projection, positive).
(9) If A is a normal matrix show that rank of A is same as the number of non-zero eigenvalues of A.
(10) Suppose N is a normal matrix. Show that

$$|\text{trace } (N)|^2 \leq \text{rank } (N).\text{trace}(N^*N).$$

(Hint: Spectral theorem and Cauchy-Schwarz inequality)