Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Home Assignment IV — Due Date : 10 May 2022
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Remark: Standard inner product is considered on $\mathbb{R}^n$ and $\mathbb{C}^n$ unless some other inner product is explicitly mentioned.

(1) (**Hadamard's inequality**). Suppose A is a positive matrix then

$$\det(A) \leq \Pi_{i=1}^{n} a_{ii}.$$

(Hint: First consider the case $a_{ii} = 1$ for every i. Use AM-GM inequality on eigenvalues. The general case should follow by considering A as a Gram matrix and suitably re-scaling the vectors.)

(2) State and present a proof of Sylvester's law of inertia (Hint: See Wikipedia and other sources).

(3) Obtain polar decompositions and singular value decompositions for following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 & 0 \\ 5 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(4) Obtain Jordan Canonical form for following matrices:

$$C = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(5) Obtain simultaneous diagonalization for the following commuting matrices:

$$E = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, F = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(6) Write E, F of previous exercise as polynomials of a single matrix G.

(7) Use Cayley-Hamilton theorem to find eigenvalues, eigenvectors and inverses of the following matrices:

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

(8) Suppose $B = [b_{ij}]$ is an $n \times n$ positive rank one matrix. Show that there exist $b_1, \ldots, b_n$ such that $b_{ij} = b_i \overline{b_j}$.

(9) Suppose A is a positive matrix. Show that A is a sum of positive rank one matrices. (Hint: Use spectral theorem).

(10) (**Schur product**) Given two square matrices $A = [a_{ij}], B = [b_{ij}]$, their Schur product $A \circ B$ is defined as the matrix $C = [c_{ij}]$, where

$$c_{ij} = a_{ij}.b_{ij}.$$

(It is the entrywise product of matrices.) Show that if A, B are positive then $C = A \circ B$ is positive. (Hint: First prove it for rank one matrices. Use exercises 8 and 9.)