Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Linear Algebra-II

Mid-term Examination I (Non-credit) — Date : 09 March 2022
Maximum marks: 100 — Time: 10.00AM-1.00PM
Instructor: B V Rajarama Bhat

(1) Let A be an $n \times n$ matrix (with $n \geq 2$) where

$$a_{ij} = \begin{cases} b & \text{if } i = j; \\ c & \text{if } i \neq j, \end{cases}$$

where b, c are fixed real numbers. (i) Write down the characteristic polynomial of A. (ii) Show that if $b \neq c$, A is invertible. (iii) If $b \neq c$, write down the inverse of A. (Hint: It has a form similar to that of A). [15]

(2) Suppose

$$P = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where A, B, are square matrices. Prove that

$$\det(P) = \det(A - B).\det(A + B).$$

[15]

(3) (i) Show that

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = 2x_1y_1 + 2x_2y_2 + x_1y_2$$

is an inner product on $\mathbb{R}^2$. (ii) Show that

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = x_1y_1 + x_2y_2 + 4x_1y_2$$

is not an inner product on $\mathbb{R}^2$. [10]

(4) (i) Show that

$$\mathcal{B} = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \}$$

is a basis for $\mathbb{R}^3$.

(ii) Apply Gram-Schmidt process on $\mathcal{B}$ and get an orthonormal basis for $\mathbb{R}^3$. [15]

(5) A unitary U is said to be a **symmetry** if $U = U^*$. Show that if P is a projection then $U := P - (I - P)$ is a symmetry. Conversely show that if U is a symmetry then $U = P - (I - P)$ for some projection P.

(6) Let $a_1, \dots, a_k$ be distinct eigenvalues of a matrix A with respective eigenvectors $w_1, \dots, w_k$:

$$A_j w_j = a_j w_j, \ 1 \leq j \leq k.$$

(i) For any fixed $i \in \{1, \dots, k\}$, consider the polynomial $p_i(x) = \Pi_{j \neq i}(x - a_j)$. Show that

$$p_i(A)w_k = 0$$

for all $k \neq i$.

(ii) Use (i) and show that $w_1, \dots, w_k$ are linearly independent. [15]

(7) Let W_1, W_2 be subspaces of a finite dimensional inner product space V and let P_1, P_2 be corresponding projections. Show that the following are equivalent: (i) W_1 and W_2 are mutually orthogonal; (ii) $P_1P_2 = P_2P_1 = 0$; (iii) $P_1 + P_2$ is a projection. [15]