LINEAR ALGEBRA -II

B V Rajarama Bhat

Indian Statistical Institute, Bangalore

Lecture 3: Leibniz formula for determinants

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- This permutation also has $2 \dashrightarrow 5 \dashrightarrow 2$, a cycle of length 2.
- It also has $4 \dashrightarrow 4$ and $6 \dashrightarrow 6$, cycles of length 1.
- For distinct $k_1, k_2, \ldots, k_r$ in $\{1, 2, \ldots, n\}$ (with $r \in \mathbb{N}$) we denote the cycle $k_1 \dashrightarrow k_2 \dashrightarrow \cdots \dashrightarrow k_r \dashrightarrow k_1$ simply as $(k_1, k_2, \ldots, k_r)$.

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- We may write down a permutation by listing the cycles it has.
- For instance, the permutation of Example 1.4, is written as $(1, 3, 7)(2, 5)(4)(6)$.

Signature of a permutation

- Definition 1.8: Let $S = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ and let σ be a permutation of S. Then the signature of σ is defined as the number
$$\epsilon(\sigma) = (-1)^{n-p}$$
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and $\sigma(j) = j$ for $j \notin \{k_1, k_2, \dots, k_r\}$.
- Therefore the signature of a cycle is defined as $(k_1, k_2, \dots, k_r) = (-1)^{n-(1+(n-r))} = (-1)^{r-1}$.

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- Permutations with signature $(+1)$ are known as even permutations and those with signature (-1) are known as odd permutations.

Permutations as products of transpositions

- Every permutation is a product of transpositions. In other words, given any permutation σ there exist transpositions $\tau_1, \tau_2, \ldots, \tau_k$ (for some $k \in \{0, 1, \ldots\}$) such that

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- Corollary 2.2: If a permutation $\tau = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k$, where $\tau_1, \ldots, \tau_k$ are transpositions then

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- Corollary 2.3: If a permutation $\tau = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_l$, where $\tau_1, \tau_2, \dots, \tau_k, \sigma_1, \sigma_2, \dots, \sigma_l$ are transpositions, then $k - l$ is even. In particular, k is odd/even if and only if l is odd/even.

Permutation matrices

- Definition 2.4: Fix $n \in \mathbb{N}$ and let σ be a permutation of $\{1, 2, \ldots, n\}$. Then the $n \times n$ matrix P^σ defined by

$$p_{ij}^\sigma = \begin{cases} 1 & \text{if } \; i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

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- We also consider the matrix P^σ as the linear transformation $x \mapsto P^\sigma x$ on $\mathbb{R}^n$. More explicitly, if $x \in \mathbb{R}^n$ has the expansion $x = \sum_{j=1}^{n} x_j e_j$ in the standard basis $\{e_1, e_2, \ldots, e_n\}$,

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$$(P^\sigma x)_i = \sum_j p_{ij}^{\sigma} x_j = x_{\sigma^{-1}(i)}.$$
- Note that $P^\sigma e_j = e_{\sigma(j)}$. Therefore P^σ just permutes the basis elements $e_1, e_2, \ldots, e_n$, sending e_j to $e_{\sigma(j)}$. Hence for any two permutations σ, τ, $P^{\tau \circ \sigma} = P^\tau . P^\sigma$.

Determinants

- Notation: Let A be an $n \times n$ matrix. Then for any $1 \leq i, j \leq n$, the matrix formed by dropping i-th row and j-th column is known as (i, j)-th minor of A and is denoted by A_{ij}.

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- Here we have written the expansion using the first column. But we could have used any row or column.

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- This suggests the following theorem.

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- To prove this theorem we use the following characterization of the determinant proved in Semester -I.

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- Then $f(A) = \det(A)$ for every $A \in M_n(\mathbb{R})$.
- The determinant satisfies (i) to (iii) and the word 'adjacent' in (iii) can be dropped. The property (ii) is known as 'multi-linearity'.

Proof of Liebniz formula

- Define $f : M_n(\mathbb{R}) \to \mathbb{R}$ by

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- If $A = I$, then, $a_{ij} = 0$ if $i \neq j$, hence

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- If σ is the identity permutation $\epsilon(\sigma) = 1$. Hence $f(I) = 1.1.\dots 1 = 1$.

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- Now

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Multi-linearity

- Consider i-th row of A. Suppose $a_{ij} = sb_{ij} + tc_{ij}$ for some real numbers s, t, b_{ij}, c_{ij}, $1 \leq j \leq n$.
- Now

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- Then it is clear that f satisfies (ii).

Continuation

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- Since for any σ in S_n, $\sigma \circ \tau \in S_n$,

$$\{\sigma \circ \tau : \sigma \in S_n\} \subseteq S_n$$

  is obvious. Now consider any permutation η in S_n. We can write η as $\sigma \circ \tau$, where $\sigma = \eta \circ (\tau)^{-1}$. This shows,

$$S_n \subseteq \{\sigma \circ \tau : \sigma \in S_n\}.$$

Continuation

- Therefore,

$$f(A) = \sum_{\sigma \in S_n} \epsilon(\sigma \circ \tau) a_{1\sigma\circ\tau(1)} a_{2\sigma\circ\tau(2)} \cdots a_{i\sigma\circ\tau(i)} \cdots a_{j\sigma\circ\tau(j)} \cdots a_{n\sigma\circ\tau(n)}.$$

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- Therefore, $2f(A) = 0$ or $f(A) = 0$. This proves (iii) and hence $f(A) = \det(A)$. ■.

Determinant of permutation matrices

- Recall that for a permutation $\sigma \in S_n$, we have defined the associated 'permutation matrix' P^σ by

$$p_{ij}^{\sigma} = \begin{cases} 1 & \text{if } \; i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

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- We see that for a term in this sum to be non-trivial we need $j = \sigma(\eta(j))$ for every j, or $\eta = (\sigma)^{-1}$.

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- An interesting problem: Show that for any $n \times n$ doubly stochastic matrix D,

$$\text{per } (D) \ge \frac{n!}{n^n}.$$

In other words, the permanent on doubly stochastic matrices