LINEAR ALGEBRA -II

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- Reference for ‘Doubly stochastic matrices’:

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- (i) Non-negative matrices and applications, R B Bapat and T E S Raghavan

Lecture 4: Determinants of partitioned matrices

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- (i) Non-negative matrices and applications, R B Bapat and T E S Raghavan
- (ii) Books on 'Markov Chains'. (For stochastic matrices).

Upper and lower triangular matrices

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$$a_{ij} = 0, \quad \forall 1 \leq i < j \leq n.$$
- So if A is upper triangular, then it has the form:
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Determinants of upper/lower triangular matrices

- Theorem 4.2: If a matrix $A = [a_{ij}]$ is upper triangular or lower triangular then the determinant of A is the product of its diagonal entries:

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

- Proof. We have Liebnitz formula:

$$\det(A) = \sum_{\sigma\in S_n} \epsilon(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}.$$

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- Alternatively, we may expand the determinant of A using first column and use induction.
- A similar proof works for lower triangular matrices through expansion using first row. ■

Partitioned vectors

- Fix $m, n \in \mathbb{N}$. Consider a vector $z \in \mathbb{R}^{m+n}$:

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m+n} \end{pmatrix}.$$

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- So we write

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$x = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}, \quad y = \begin{pmatrix} z_{m+1} \\ z_{m+2} \\ \vdots \\ z_{m+n} \end{pmatrix}.$$

Continuation

- Conversely, given any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we get a vector $z \in \mathbb{R}^{m+n}$ as
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- Conversely, given any $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we get a vector $z \in \mathbb{R}^{m+n}$ as
$$z = \begin{pmatrix} x \\ y \end{pmatrix}.$$
- So in a way, we can think of $\mathbb{R}^{m+n}$ as constructed out of $\mathbb{R}^m$ and $\mathbb{R}^n$. We say that $\mathbb{R}^{m+n}$ is direct sum of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Partitioned matrices or block matrices

- Now consider a matrix $P = [p_{ij}]_{1 \le i,j \le (m+n)}$ considered as a linear map on $\mathbb{R}^{m+n}$.

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- We partition P as
$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$
where $A_{m\times m}, B_{m\times n}, C_{n\times m}, D_{n\times n}$ are given by
$$A = \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{bmatrix}, \quad B = \begin{bmatrix} p_{1(m+1)} & \cdots & p_{1(m+n)} \\ \vdots & \ddots & \vdots \\ p_{m(m+1)} & \cdots & p_{m(m+n)} \end{bmatrix}.$$

Continuation

- $$C = \begin{bmatrix} p_{(m+1)1} & \cdots & p_{(m+1)m} \\ \vdots & \ddots & \vdots \\ p_{(m+n)1} & \cdots & p_{(m+n)(m)} \end{bmatrix},$$

Continuation

- $$C = \begin{bmatrix} p_{(m+1)1} & \cdots & p_{(m+1)m} \\ \vdots & \ddots & \vdots \\ p_{(m+n)1} & \cdots & p_{(m+n)(m)} \end{bmatrix},$$

- $$D = \begin{bmatrix} p_{(m+1)(m+1)} & \cdots & p_{(m+1)(m+n)} \\ \vdots & \ddots & \vdots \\ p_{(m+n)(m+1)} & \cdots & p_{(m+n)(m+n)} \end{bmatrix}$$

The action of partitioned matrices on vectors

- Under notation as above, with

$$Pz = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} Ax + By \\ Cx + Dy \end{array} \right).$$

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- Note that $A : \mathbb{R}^m \to \mathbb{R}^m$, $B : \mathbb{R}^n \to \mathbb{R}^m$, $C : \mathbb{R}^m \to \mathbb{R}^n$ and $D : \mathbb{R}^n \to \mathbb{R}^n$.

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- In other words, the multiplication is like the usual matrix multiplication.
- Proof. The proof is by direct multiplication.

Continuation

- For instance, for $1 \leq i, j \leq m$,

$$\begin{aligned}(PQ)_{ij} &= \sum_{k=1}^{m+n} p_{ik} q_{kj} = \sum_{k=1}^{m} p_{ik} q_{kj} + \sum_{k=m+1}^{m+n} p_{ik} q_{kj} \\ &= (AE)_{ij} + (BG)_{ij}\end{aligned}$$

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- More generally, if $P = [A_{ij}], \quad Q = [B_{kl}]$ are partitioned matrices, with matching orders, then PQ is a partitioned matrix $[C_{ij}]$ with

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- Here, for the matrix multiplication to be meaningful, it is necessary that for fixed i, k, j, if the order of A_{ik} is $a \times b$ then the order of B_{kj} should be $b \times c$ for some c. This is what we mean by ‘matching orders’.

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- More generally, if we have a partitioned matrix

$$P = [A_{ij}]$$

  then

$$P^t = [(A_{ji})^t].$$

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- In a similar way one can define block lower triangular matrices.

Determinants of block upper triangular matrices

- Theorem 4.4: Consider a block upper triangular matrix

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where A, D are square matrices and $C = 0$. Then

$$\det(P) = \det(A).\det(D).$$

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$$\det(P) = \sum_{\sigma \in S_{m+n}} \epsilon(\sigma) P_{1\sigma(1)} P_{2\sigma(2)} \ldots P_{(m+n)\sigma(m+n)}.$$

- Now $C = 0$, means that $P_{j\sigma(j)} = 0$ if $(j, \sigma(j))$ are such that $(m+1) \leq j \leq (m+n)$ and $1 \leq \sigma(j) \leq m$.

Continuation

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- Such permutations are precisely permutations of the form $\tau \circ \eta$ where τ is permutation of $\{1, 2, \ldots, m\}$ considered as a permutation of $\{1, 2, \ldots, (m+n)\}$ by taking $\tau(j) = j$ for $j \in \{m+1, \ldots (m+n)\}$ and η is a permutation of $\{m+1, \ldots, m+n\}$ extended to $\{1, \ldots, (m+n)\}$ by taking $\eta(j) = j$ for $1 \leq j \leq m$.

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- Note that the signature of a permutation does not change by considering such extensions.

Continuation

- Then it is clear that,

$$\begin{aligned} & \det(P) \\ = & \sum_{\tau,\eta} \epsilon(\tau).\epsilon(\eta) p_{1\tau(1)} \cdots p_{m\tau(m)} \cdot p_{m+1\eta(m+1)} \cdots p_{m+n\eta(m+n)} \\ = & \det(A). \det(D). \blacksquare \end{aligned}$$

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- Now by mathematical induction the determinant of a block upper triangular matrices (with square blocks on the diagonal) is the product of the determinants of diagonal blocks. That is,

$$
\det\left(\begin{bmatrix} P_{11} & P_{12} & P_{13} & \dots & P_{1n} \\ 0 & P_{22} & P_{23} & \dots & P_{2n} \\ 0 & 0 & P_{33} & \dots & P_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & P_{nn} \end{bmatrix}\right) = \det(P_{11}) \dots \det(P_{nn}).
$$

if $P_{11}, P_{22}, \dots, P_{nn}$ are square blocks.

Inverses of 2×2 upper triangular matrices.

- Theorem 4.5: Consider a block upper triangular matrix

$$P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

where A, D are square matrices and $C = 0$. Then P is invertible if and only if A and D are invertible and in such a case,

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- From the formula $\det(P) = \det(A).\det(D)$, we know that if P is invertible, then $\det(A)$ and $\det(D)$ are non-zero and hence A, D are invertible.
- The formula for the inverse can be confirmed by verifying:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} . \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

A special case

- Corollary 4.6: For any matrix B,

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^n = \begin{bmatrix} I & nB \\ 0 & I \end{bmatrix}$$

for every $n \in \mathbb{Z}$.

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- Proof: The result is clear for $n = 0, 1$. Now verify the formula for $n \in \mathbb{N}$ by induction. Taking inverses we have the result for all $n \in \mathbb{Z}$.

- This is actually a consequence of

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & B + C \\ 0 & I \end{bmatrix}.$$

  The matrix product becomes simple addition here.

A special case

- Corollary 4.6: For any matrix B,

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}^n = \begin{bmatrix} I & nB \\ 0 & I \end{bmatrix}$$

for every $n \in \mathbb{Z}$.

- Proof: The result is clear for $n = 0, 1$. Now verify the formula for $n \in \mathbb{N}$ by induction. Taking inverses we have the result for all $n \in \mathbb{Z}$.
- This is actually a consequence of

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & B + C \\ 0 & I \end{bmatrix}.$$

The matrix product becomes simple addition here.

- END OF LECTURE 4.